FINITE DIMENSIONAL LIE AND ASSOCIATIVE ALGEBRAS

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1 INTRODUCTION

1.1 Definition of a Lie Algebra

Dfn 1.1: A **Lie Algebra** is a k-vector space and a bilinear operation $[\cdot,\cdot]: L \times L \to L$ satisfying

(1) $[x, x] = 0 \quad \forall x \in L$

(2) Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

if char $k \neq 2$, then $1 \Leftrightarrow [x,y] = -[y,x]$.

$[x+y, x-y] = -[x-y, x+y]$

$\Rightarrow [x,x] + [x,-y] + [y,x] + [y,-y] = -\left([x,x] + [x,y] + [-y,x] + [-y,y]\right)$

$\Rightarrow [x,x] + [x,-y] + [y,x] + [y,-y] = -[x,x] + [y,x] + [x,-y] + [y,-y]$

$\Rightarrow [x,x] = -[x,x]$: char $k \neq 2 \Rightarrow [x,x] = 0$

char $k = 2 \Rightarrow$ can't say anything further.

Motivation: Groups $\longrightarrow$ symmetries

Lie grp. $\longrightarrow$ infinitesimal symmetries.

Exm: $G = GL_n(\mathbb{R})$ is a Lie group. From this, we have an associated Lie Algebra given by the tangent space at identity $T_1 G$. $T_1 G \cong M_n(\mathbb{R})$ (analytic manifold)

exp : nhood of 0 in $M_n(\mathbb{R})$ $\to$ nhood of 1 in $GL_n(\mathbb{R})$

inverse: log

$\exp A \exp B = \exp(\mu(A,B))$

where $\mu(A,B) = A + B + \frac{1}{2}[A,B] + O(\geq 3)$

where $[A,B] = AB - BA \leftarrow$ matrix mult.

(in general, given an associative algebra R, we can define a lie bracket on R by

$[R,S] = RS - SR$ (alg mult).

FACTS

(1) first approximation to the group operation is addition in $T_1 G$.

(2) If $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ (group commutator)

the Lie Bracket $[A,B]$ is the first approximation in $T_1 G$ of commutator $[\exp A, \exp B]$ in G.

(3) Jacobi identity arises from the associativity of the group operation

Exm $G = GL_n(\mathbb{C})$ is an algebraic group (complex alg variety w/ continuous operation)

Then $T_1 G \cong M_n(\mathbb{C})$, similarly define $[,] \Rightarrow$ complex Lie Algebra

1.2. Simple / Semisimple Lie Algebras

Dfn 1.2 (a) A lie subalgebra J of L is a K-subspace of L s.t $\forall\ x,y \in J$, $[x,y] \subset J$.

(b) a (lie) ideal J of L is a K-subspace s.t $[x,y] \in J\ \forall\ x \in J$, $y \in L$.
note: dfn is symmetric actually in x and y.

Dfn 1.3 (a) L is semisimple if $R(L) = 0$ (radical, coming soon!) and in general, $L/R(L)$ is semisimple.

(b) L is simple $\Leftrightarrow$ the only ideals are 0 and L. also $[L,L] \neq 0$, to avoid 1-dimensional case.

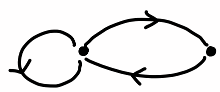
FACT] fin dim Lie alg, semisimple: direct prod of simple ones

Will concentrate on classifying simple fin. dim complex lie alg:
↳ classification of "finite root systems"

root system: collection of well behaved combinatorial data.
↳ has a symmetry group called the weyl group, which is an example of a coxeter group.

Root systems also arise in the representation of quivers.

A quiver is a directed graph (vertices and directed arrows)
can have multiple directed edges and loops:



A representation of a quiver:
associate a vector space to each vertex, and a linear map to each directed edge (in the given direction).

Associated with a quiver you have a path algebra:
associative algebra: basis corresponding to paths. (concatenating them)

point is, the modules over path algebra correspond to representations of quiver.

Consider indecomposable representations (i.e. ones that cannot be expressed as a direct sum of two nontrivial representations).

QUESTION] Which quivers only have finitely many indecomposable representations?
Answer: Gabriel: classify root systems.

FINALLY FOR TODAY: run through basics on finite dim associative algebras.
define radical, a canonical ideal.

Jacobson radical, R semisimple $\Leftrightarrow J(R) = 0$
$R/J(R)$ always semisimple

for finite dim. $\Rightarrow$ Artin - Wedderburn
↳ semisimple K-alg are direct products of simple ones
" simple ones $\cong M_n(D)$ D division algebra.

2 LIE ALGEBRAS

2.1 Associative Algebras

recall: $Z(R) = \{ x \in R : \forall y \in R,\ xy = yx \}$.

Dfn (associative K-algebra) ring with 1 and $\phi: K \to R$ a ring homomorphism $1 \mapsto 1$ and want that $\phi(K) \subseteq Z(R)$ (centre of R)

Idea: take an associative K-algebra, give it a Lie Bracket and turn it into a Lie algebra.

Then R is a Lie Algebra via $[r,s] = rs - sr$.
In particular, $M_n(K)$ is a Lie Algebra associated with GL_n.

Check: $[r,r] = rr - rr = 0$
$[r+x, s] = (r+x)s - s(r+x) = (rs - sr) + (xs - sx) = [r,s] + [x,s]$ + similarly $[r, s+y] = [r,s] + [r,y]$.
$[x,[y,z]] = [x, yz - zy] = [x,yz] - [x,zy] = xyz - yzx - xzy + zyx$
But $\circlearrowleft (xyz - yzx - xzy + zyx) = 0$.

2.2. Classic Examples

(1) matrices of trace $= 0 =: \mathfrak{sl}_n$ is associated with SL_n.

$SL_n = \{ n \times n$ matrices w/ determinant $1\}$

$\mathfrak{gl}_n \cong M_n(K)$

e.g. $\mathfrak{sl}_2$ standard notation: $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
note that $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

(2) $\mathfrak{so}_n$ = skew symmetric $n \times n$ matrices, associated with special orthogonal group SO_n.

e.g. $n = 3$, $\mathfrak{so}_3$: $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, $[A_3, A_1] = A_2$.

(3) $\mathfrak{sp}_{2n}$ = contains matrices associated with symplectic group Sp_{2n} of matrices that preserve a non-degenerate, skew-symmetric product on K^{2n}.

e.g. let's say the skew-symmetric form is represented by $J = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & ⋰ & \\ -1 & & \end{smallmatrix} & 0 \end{array}\right)_{2n \times 2n}$

Then $\mathfrak{sp}_{2n}$ consists of matrices X s.t. $XJ + JX^t = 0$

(alternative formulation: take $J = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right)$)

these matrices are of form $\left(\begin{array}{c|c} A & B \\ \hline C & -A^T \end{array}\right)$, B and C symmetric.

dimension of $\mathfrak{sp}_{2n}$ is $2n^2 + n$

For this formulation of J you get $\left(\begin{array}{c|c} A & B \\ \hline C & A^T \end{array}\right)$, B and C skew-symmetric.

(4) $\mathfrak{b}_n$ **Borel subalgebra** of $\mathfrak{gl}_n$ of upper triangular matrices associated with the Borel subgroup of GL_n consisting of invertible upper triangular matrices.

(5) $\mathfrak{n}_n$ consists of strictly upper triangular matrices associated with the group of upper triangular matrices with 1's on the diagonal. ($\mathfrak{n}$ denotes nilpotent).

2.3. Derivations

Given an associative algebra R, we can define a Lie subalgebra of $\mathrm{End}_K(R)$:

Dfn 2.1 a linear map $D: R \to R$ is a **derivation** if $D(rs) = D(r)s + rD(s)$

Leibniz property

ring product

not multiplication but composition, since in End. ring op. is comp, because identity is id. map $1: x \mapsto x$, $x \mapsto 1$ is not a homo of R b.c. it would send $0 \mapsto 1$.

$\{\text{derivations } R \to R\}$ forms a lie subalgebra of $\mathrm{End}_K(R)$ $\to$ form Lie algebra via $[f,g] = f \circ g - g \circ f$

Closed under $[\cdot,\cdot]$: Let $D_1, D_2 \in \mathrm{Der}_K(R)$. WTS that $[D_1, D_2] \in \mathrm{Der}_K(R)$. Now, $[D_1, D_2] = D_1 D_2 - D_2 D_1$.
Then for $r, s \in R$,
$(D_1 D_2)(rs) = D_1(D_2(rs)) = D_1(D_2(r)s + rD_2(s)) = (D_1 D_2)(r)s + D_2(r)D_1(s) + D_1(r)D_2(s) + r(D_1 D_2)(s)$
So that $(D_1 D_2)(rs) - (D_2 D_1)(rs) = (D_1 D_2 - D_2 D_1)(r)s + r(D_1 D_2 - D_2 D_1)(s) = [D_1, D_2](r)s + r[D_1, D_2](s)$.
I.e $[D_1, D_2]$ is a derivation.

e.g. $\mathrm{Der}(K[x]) = \{ f(x)\frac{d}{dx} : f(x) \in K[x] \}$

equivalently: $D([r,s]) = [D(r), s] + [r, D(s)]$

e.g. $\mathrm{Der}(K[X, X^{-1}])$ = **Witt algebra** closely related to Virasoro lie algebra

notice: no need for R to be commutative. Geometrically if R is a coordinate ring, then the derivations correspond to vector fields.

Dfn 2.2 An **inner derivation** of R is of the form $R \to R$, $s \mapsto [r,s]$ for some $r \in R$.

$\mathrm{Innder}(R) = \{\text{inner derivations}\}$ forms a lie ideal in $\mathrm{Der}(R)$.

$\mathrm{Innder}(R) \trianglelefteq \mathrm{Der}(R)$. Let $D_1 \in \mathrm{Innder}(R)$ and $D_2 \in \mathrm{Der}(R)$. WTS that $[D_1, D_2] \in \mathrm{Innder}(R)$.
Now, $D_1(s) = [r,s]$ for some $r \in R$. Hence,

$$\begin{aligned}
[D_1, D_2](q) &= D_1 D_2(q) - D_2 D_1(q) = [r, D_2 q] - D_2([r,q]) \\
&= [r, D_2 q] - ([D_2(r), q] + [r, D_2(q)]) \\
&= [D_2(r), q] \in \mathrm{InnDer}(R).
\end{aligned}$$

e: **Rm**

(1) Further, $\mathrm{Der}(R) / \mathrm{Innder}(R) \cong HH^1(R,R)$ (1st Hochschild cohomology group of R).

(2) If R is commutative, then $\mathrm{Innder}(R) = 0$

R commutative, then $[x,y] = xy - yx = xy - xy = 0$
$\Rightarrow$ only map in Innder is 0 map

(3) Lie algebras arise from considering derivations of other algebraic structures.

2.4. Representations

Dfn 2.3 (a) A lie algebra homomorphism $\rho: L_1 \to L_2$ is a K-linear map satisfying

$$\rho([x,y]) = [\rho(x), \rho(y)]$$

(b) A linear representation of a lie algebra L is a lie algebra homomorphism $\rho_V : L \to \mathrm{End}(V)$

into some vector space you choose!

If $U \le V$ and $\rho_V(L)(U) \le U$, then there is a subrepresentation

$$\rho_U : L \to \mathrm{End}(U)$$
$$\rho_U(x)(u) := \rho_V(x)(u) \qquad u \in U,\ x \in L$$

(c) an irreducible representation is one where the only such U are 0 and V

Exm (1) $\mathrm{ad}_L : L \to \mathrm{End}(L)$ (adjoint representation)

$$x \mapsto \underset{[x,-]}{\mathrm{ad}(x)} : L \to L\ ;\ y \mapsto [x,y].$$

is a lie alg. hom. because of Jacobi.

$\mathrm{ad}(0) = [0,-] = 0$ map:

$[0,y] = [0\cdot x, y]$ for any $x \in L$

$= 0[x,y$

$$\begin{aligned}
\mathrm{ad}([x,y])(z) = [[x,y],z] &= [x,[y,z]] + [y,[z,x]] \\
&= \mathrm{ad}(x)([y,z]) + \mathrm{ad}(y)([z,x]) \\
&= \mathrm{ad}(x)([y,z]) - \mathrm{ad}(y)([x,z]) \\
&= \mathrm{ad}(x)(\mathrm{ad}(y)(z)) - \mathrm{ad}(y)(\mathrm{ad}(x)(z)) \\
&= [\mathrm{ad}(x), \mathrm{ad}(y)](z).
\end{aligned}$$

Dfn 2.4 : The centre of L $= \{x : [x,y] = 0 \ : \forall y \in L\}$.
$= \ker(\mathrm{ad}_L)$.

If ad_L is injective, then L embeds in $\mathrm{End}(L)$, and so L may be regarded as a lie subalgebra of $\mathrm{End}(L)$ in this case.

Thm (of Ado) : if $\mathrm{char}\, K = 0$, a finite dimensional L can always be embedded in some $\mathrm{End}(V)$.

In fact, it's also true in $\mathrm{char}\, K = p$ (Iwosawa) (much harder).

Exm (2) let $K = \mathbb{R}$. $\mathbb{R}^3$ is a lie algebra using the vector product

standard basis e_1, e_2, e_3. Then
$$e_1 \times e_2 = e_3$$
$$e_2 \times e_3 = e_1$$
$$e_3 \times e_1 = e_2$$

and $\mathrm{ad}_L : L \to \mathrm{End}(L) \cong M_3(\mathbb{R})$
$$e_i \mapsto A_i \in \mathfrak{so}_3$$

$\Rightarrow \ker(\mathrm{ad}_L) = 0, \quad \mathrm{Im}(\mathrm{ad}_L) = \mathfrak{so}_3.$

Thus, $(\mathbb{R}^3, \text{vector product}) \cong \mathfrak{so}_3(\mathbb{R})$.

Representation: lie algebra homomorphism $\rho : L \to End(V)$.

sl_2 = 2x2 matrices with trace = 0
$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

Example 2: Some representations of sl_2.

Take $K[X,Y]$: polynomial algebra in 2 variables, and construct a linear map $sl_2 \to Der(K[X,Y])$ by sending

$e \mapsto X\, \partial/\partial Y$
$f \mapsto Y\, \partial/\partial X$
$h \mapsto X\, \partial/\partial X - Y\, \partial/\partial Y$

Claim: this is a lie algebra homomorphism.

$[\rho(e), \rho(f)] = [X \partial/\partial Y, Y \partial/\partial X]$
$= X \partial/\partial Y (Y \partial/\partial X) - Y \partial/\partial X (X \partial/\partial Y)$
$= X \partial/\partial X - Y \partial/\partial Y = \rho(h)$ ✓

$[\rho(h), \rho(e)] = [X \partial/\partial X - Y \partial/\partial Y, X \partial/\partial Y]$
$= X \partial/\partial Y - - X \partial/\partial Y = 2X \partial/\partial Y = 2\rho(e)$ ✓

$[\rho(h), \rho(f)] = [X \partial/\partial X - Y \partial/\partial Y, Y \partial/\partial X]$
$= - Y \partial/\partial X - Y \partial/\partial X = -2\rho(f)$ ✓

Define V_n = span of monomials of total degree n
= Space of homogeneous polynomials of degree n

$\rho(sl_2)(V_n) \subseteq V_n$ $\quad$ $\dim V_n = n+1$ $\quad$ get subrep. ρ_n
and $V = \bigoplus V_n$

Consider when $n=1$. Then

$$\left.\begin{aligned} sl_2 &\to End(V_1) \cong M_2(K) \\ e &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \text{Canonical representation}$$

Consider when $n=2$. Then $sl_2 \to End(V_2) \cong M_3(K)$ is the adjoint representation ad_{sl_2} (check)

Lem 2.5: $\rho_n : sl_2 \to End(V_n)$ is irreducible for all n.

pf: Let $0 \neq U \leq V_n$ such that $\rho(sl_2) \subseteq U$. We want to show $U = V_n$. So, take a nonzero homogeneous polynomial of degree n:

$$\sum_{i+j=n} \lambda_{ij} X^i Y^j$$

Apply $X \partial/\partial Y$ to this polynomial sufficiently many times to get a nonzero multiple of X^n. Hence $X^n \in U$. Now apply $Y \partial/\partial X$ repeatedly to get a nonzero multiple of other monomials $X^i Y^j$ with $i+j=n$. Thus, $V_n \subseteq U$, so $U = V_n$. Hence the representation is irreducible. □

Terminology: representation refers to the map, but in practice, many refer to V as a representation (the vector space). Brookes tends to refer to V as an L-module, by analogy with usage in ring theory, and simple L-modules corresponding to irreducible representations

Warning: definition of a simple Lie algebra is non-standard! Most people don't allow the one-dimensional Lie algebra to be simple. With Brookes definition, we can observe

Observation 2.6: L is a simple Lie algebra $\Leftrightarrow$ ad_L is irreducible.

Cor 2.7 (of 2.5): sl_2 is a simple Lie algebra □

pf: we've seen that ad_{sl_2} is irreducible in 2.5 ($n=2$ case)

2.5. Soluble Lie Algebras

Dfn 2.8: An Abelian Lie algebra L if $[x,y]=0\ \forall\, x,y\in L$.

Dfn 2.9: The derived series of L is defined intuitively:

$$L^{(0)}=L\ ,\quad \underset{\text{derived subalgebra}}{L^{(1)}}=[L,L]=\operatorname{span}\{[x,y]:x,y\in L\}$$

$$L^{(i)}=[L^{(i-1)},L^{(i-1)}]\qquad i\geqslant 2.$$

Dfn 2.10: L is Soluble (solvable) if $L^{(r)}=0$ for some r. The least such r is known as the derived length of L.

Note: each $L^{(i)}$ is an ideal of L

Suffices to show $L^{(i)}$ is an ideal of L. if $a\in L^{(1)}$, then $\exists\ x,y\in L$ s.t $a=[x,y]$. Let $z\in L$. Then $[[x,y],z]\in L^{(1)}$ since $[x,y],z\in L$. Let i case hold. $L^{(i+1)}=[L^{(i)},L^{(i)}]$. if $a\in L^{(i+1)}$, then $[a,z]\in L^{(i+1)}$, since say $a=[x,y]$, $x,y\in L^{(i)}$ and by Jacobi,

$$[[x,y],z]=[\underset{\in L^i}{y},\underset{\in L^i}{[z,x]}]+[\underset{\in L^i}{x},\underset{\in L^i}{[y,z]}]$$

by induction

$\Rightarrow [[x,y],z]\in[L^{(i)},L^{(i)}]=L^{(i+1)}$.

$\Rightarrow L^{(i+1)}$ is an ideal of L.

Non-zero Abelian Lie algebras are precisely those of derived length 1.

Rem If J is an ideal of L, then L/J has the structure of a lie algebra via $[J+x,J+y]=[x,y]+J$.

Lemma 2.11

(1) Subalgebras and quotients of Soluble lie algebras are soluble

(2) If J is an ideal of L, then

$$L \text{ is Soluble} \iff J \text{ and } L/J \text{ are soluble}$$

Pf: (1) Suppose $L^{(r)}=0$, and let I be a subalgebra. Consider that $I\subseteq L$, so $[I,I]\subseteq[L,L]$, and so going down the line we see that $I^{(r)}\subseteq L^{(r)}$. But if $L^{(r)}=0$, then $\Rightarrow I^{(r)}=0$ so I is soluble.

Let $J\trianglelefteq L$ be an ideal. If $L^{(r)}=0$, then $(L/J)^{(r)}=L^{(r)}+J=0+J=J\Rightarrow (L/J)^{(r)}=0$.

(2) Suppose $(L/J)^{(r)}=0$ and $J^{(s)}=0$ for some $r,s\in\mathbb{Z}$. Notice that $(L/J)^{(i)}=L^{(i)}/J$ since we have that $[x+J,y+J]=[x,y]+J$. Hence, if $(L/J)^{(r)}=0$, $\iff L^{(r)}/J=0$, so that actually $L^{(r)}$ is a subalgebra of J. But $J^{(s)}=0$, so really $L^{(rs)}=0\Rightarrow L$ is soluble. $\square$

Rem: (2) can partly be rexpressed as: if I is a soluble ideal of L such that L/I is soluble, then L itself is soluble.

Example: (1) let L be any 2-dimensional Lie algebra.

Case ⓐ $[x,y]=0$ $\forall x,y \in L$ and L is abelian

ⓑ $\exists$ $x,y \in L$ s.t $[x,y] \neq 0$.

However, $\{x,y\}$ form a basis then of L.

$L^{(1)}$ is span of $\cancel{[x,x]}^{=0}$, $[x,y]$, $\cancel{[y,y]}^{=0}$

$\Rightarrow$ $L^{(1)}$ is 1-dimensionally spanned by $[x,y]$.

$[x,x]=0$

But 1-dimensional Lie algebras must be abelian (axiom 1 of Lie algebras)

So we get a derived series $L \supseteq L^{(1)} \supseteq L^{(2)}=0$

To summarize, in case ⓐ derived length $=1$, case ⓑ derived length $=2$. In both cases, L is soluble.

$\square$

Exercise: classify 3-dimensional Lie algebras.

(2) The Lie algebra so_3 is not soluble. Consider that we have a basis x,y,z, with $[x,y]=z$, $[y,z]=x$, $[z,x]=y$. Hence, $L^{(1)}=L$.

Lemma 2.12: The sum of two soluble ideals is a soluble ideal.

pf: Let J_1, J_2 be soluble ideals. Then J_1+J_2 is an ideal, and

J_1+J_2/J_1 is an ideal of L/J_1 and is the image of J_2 under the canonical map $L \to L/J_1$. Hence J_1+J_2/J_1 is soluble. Now use 2.11 to see that J_1+J_2 is soluble.

$\square$

Let L be any arbitrary Lie algebra and S a maximal soluble ideal. If I is any other soluble ideal of L, then $S+I$ is soluble. By maximality, $\Rightarrow$ $S+I=L$, or $I \subseteq S$. So S is actually the unique maximal soluble ideal of L. This motivates the following definition:

Dfn 2.13: The radical $R(L)$ of finite dimensional Lie algebra L is the maximal soluble ideal. It is the sum of all the soluble ideals.

Recall definition: L is semisimple $\Leftrightarrow$ $R(L)=0$

If L is a finite, soluble Lie algebra, then $R(L)=L$.

Exm: Simple Lie Algebras are semisimple

> Suppose that L is simple. Then $L^{(i)}$ is an ideal of L $\forall i$, so for each i, $L^{(i)}=0$ or L. If $L^{(i)}=L$ $\forall i$, then L is not soluble, and the only other ideal of L is 0, so we must have $R(L)=0 \Rightarrow L$ is semisimple. If $L^{(i)}=0$ for some (minimal) i, then $[L^{(i-1)}, L^{(i-1)}]=L^{(i)}=0$. But by minimality, $L^{(i-1)}=L$, so that $[L,L]=0$. But this contradicts the definition of a simple Lie algebra. So actually more generally, simple Lie algebras are not soluble.

$\square$

Note: In general that $R(L/R(L))=0$ since a soluble ideal of $L/R(L)$ would pull back to an ideal of L containing $R(L)$ and 2.11 would show that this was itself soluble and hence contained in $R(L)$. Thus, $L/R(L)$ is semisimple.

Thm 2.14 (Levi, proof omitted). If char $K = 0$ and L is finite dimensional, then there exists a Lie subalgebra L_1 such that $L_1 \cap R(L) = 0$, and $L = L_1 + R(L)$

Hence $L_1 \cong L/R(L)$ is semisimple.

Dfn 2.15: This is the Levi decomposition, and L_1 is the Levi factor (/ subalgebra).

Rem: This does NOT NECESSARILY apply in char $K = p$ or for infinite dimensional Lie algebras.

Example 1: $L = gl_2$, then $R(L) = Z(L) = \{$ scalar matrices $\lambda I\}$

then $L = sl_2 + R(L)$

sl_2 is a simple algebra, and therefore is semisimple. Thus, sl_2 is a levi subalgebra

Example 2:

$$L = \left(\begin{array}{c|c} sl_2 & \begin{matrix} * & * \\ * & * \end{matrix} \\ \hline 0 & sl_2 \end{array}\right) \subseteq gl_4$$

Then $R(L) = \left(\begin{array}{c|c} 0 & \begin{matrix} * & * \\ * & * \end{matrix} \\ \hline 0 & 0 \end{array}\right)$ Check: soluble ideal

Levi subalgebra: $L_1 = \left(\begin{array}{c|c} sl_2 & 0 \\ \hline 0 & sl_2 \end{array}\right) \simeq sl_2 \times sl_2$ which is semisimple.

rem: a soluble ideal of $sl_2 \times sl_2$ would project to each component to give a soluble ideal of sl_2 (and hence 0).

2.6. Nilpotent Lie Algebras

Dfn 2.16: The lower central series of L is defined inductively:

$$L_{(1)} = L, \quad L_{(i+1)} = [L_{(i)}, L]$$ (take span of these elements) $i \geq 2$.

Note: (1) $L_{(i)}$ are ideals of L

(2) Counting starts at 1.

$L_{(2)} = [L_{(1)}, L] = [L, L]$. Then for any $x \in L_{(2)}$, $y \in L$, of course $x \in L \Rightarrow [x,y] \in [L,L] = L_{(2)}$

So $L_{(2)}$ is an ideal. Let $L_{(i-1)}$ be an ideal. Let $x \in L_{(i)}$, $y \in L$.

Then $[x, y] = [[a,b], y]$ for $a \in L_{(i-1)}, b \in L$

By Jacobi, $= [[y,a],b] + [[b,y],a]$

$\in [L_{(i-1)}, L] + [L_{(i-1)}, L] = L_{(i)}$. $\square$

We say that L is nilpotent if $L_{(c)} = 0$ for some c, and the nilpotency class of L is the least such c.

Lemma 2.17: $L^{(n)} \leq L_{(2^n)}$ $\forall n$

pf: exercise.

Proposition (page 12 of Humphrey's): Let L be a Lie Algebra.

a) if L is nilpotent, then so are all subalgebras and homomorphic images of L

b) if $L/Z(L)$ is nilpotent, then so is L.

c) if L is nilpotent and nonzero, then $Z(L) \neq 0$.

Pf. (a) if L is nilpotent and $\rho: L \to M$ is a (wlog surjective) lie homomorphism, then $\exists\, c \in \mathbb{N}$ such that $L_{(c)} = 0$. But if $L_{(c)} = 0$, then

$$\rho(L_{(c)}) = \rho[L_{(c-1)}, L] = [\rho(L_{(c-1)}), \rho(L)] = [\rho(L_{(c-1)}), M]$$

Continuing onwards, it follows that actually $\rho(L_{(c)}) = M_{(c)}$. But $\rho(L_{(c)}) = \rho(0) = 0 \Rightarrow M_{(c)} = 0$.
Hence M is nilpotent, with nilpotency class $\leq c$.

Now suppose L nilpotent and $J \leq L$ is a subalgebra. Then it's easy to see $J_{(i)} \subseteq L_{(i)}$ $\forall i$.
But L nilpotent $\Rightarrow L_{(c)} = 0$ for some $c \in \mathbb{N}$.

$$\Rightarrow J_{(c)} \subseteq L_{(c)} = 0$$

$\Rightarrow J$ is nilpotent, with nilpotency class $\leq c$.

(b) Let $L/Z(L)$ be nilpotent. Then $\exists\, c \in \mathbb{N}$ s.t $(L/Z(L))_{(c)} = 0$. But by dfn of the Lie Bracket on quotients,

$$(L/Z(L))_{(c)} = L_{(c)}/Z(L)$$

(using fact that centre is an ideal, and anything bracketed with centre is zero)

This says that $L_{(c)} \subseteq Z(L)$. But $Z(L) = \{x \in L : [x,y] = 0 \ \forall y \in L\}$. So all we need to do to show L is nilpotent is take the Lie Bracket with one more L:

$$L_{(c+1)} = [L_{(c)}, L] \subseteq [Z(L), L] = 0$$

$\Rightarrow L$ is nilpotent, with nilpotency class $\leq c+1$.

(c) If L is nilpotent, then $\exists\, c \in \mathbb{N}$ s.t $L_{(c)} = 0$, i.e. that $[L_{(c-1)}, L] \neq 0$. Supposing that c is minimal, $L_{(c-1)} \neq 0$. By definition of $Z(L)$, $\Rightarrow 0 \neq L_{(c-1)} \subseteq Z(L)$. So $Z(L) \neq 0$. $\square$

Example: $\mathfrak{n}_n :=$ strictly upper triangular $n \times n$ matrices $\subseteq \mathfrak{gl}_n$

e.g. $\mathfrak{n}_3$ Heisenberg Lie Algebra

basis: $\underset{X}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \underset{Z}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \underset{Y}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$

Then $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$.

$Z(L) = Z$. The Heisenberg lie algebra is nonabelian, and has nilpotency class 3.

Example 3: (soluble but not nilpotent):
$\mathfrak{b}_n$ = Borel = upper triangular $n \times n$ matrices.

$$\mathfrak{b}_n^{(1)} = \mathfrak{n}_n, \quad \mathfrak{b}_n \text{ is soluble but not nilpotent}$$

2.7. Lie and Engel's Theorems

Thm 2.18: (Lie) For algebraically closed k, $\mathrm{char}\, k = 0$. Suppose $L \leq \mathrm{End}(V)$ with $\dim V < \infty$. Suppose L is soluble. Then $\exists\, v \in V$, $v \neq 0$, such that $x(v) = \lambda v$ for all $x \in L$.

This is saying that v is a common eigenvector.

Thm 2.19: (Engel) Suppose $L \leq \mathrm{End}(V)$ is a Lie subalgebra, $\dim V < \infty$, and every element of L is a nilpotent endomorphism (i.e. $\forall\, x \in L$, $\exists\, a \in \mathbb{N}$ s.t $x^a = 0$). Then $\exists\, v \neq 0$, $v \in V$ such that $x(v) = 0$ $\forall x \in L$.

An easy induction shows we can represent L by strictly upper triangular matrices. Thus $L \leq \mathfrak{n}_n$. In particular, L is a nilpotent Lie Algebra

Using 2.18 and an easy inductive argument we can show that there is a chain of subspaces

$$0 = V_0 \lneq V_1 \lneq \cdots \lneq V_n = V$$

with $\dim V_i = i$, and $L(V_i) \leq V_i$. Such a chain is called a maximal flag.

If we take a basis of V so that $V_i = \langle e_1, \ldots, e_i \rangle$, then we get that L is represented by upper triangular matrices and so L can be regarded as a Lie subalgebra of $\mathfrak{b}_n$.

3 INVARIANT FORMS + CARTAN - KILLING CRITERION.

3.1 Invariant forms

Dfn 3.1: A symmetric bilinear form $\langle \cdot,\cdot \rangle : L \times L \to K$ is **invariant** if $\langle [x,y], z \rangle = \langle x, [y,z] \rangle$.

Dfn 3.2: (a) if $\rho : L \to \mathrm{End}(V)$ with $\dim V < \infty$ is a representation, then

composition of endomorphisms.

$$\langle x,y \rangle_\rho = \mathrm{Tr}\,(\rho(x)\rho(y))$$

is the **trace form of ρ.**

(b) The trace form of the adjoint representation (when $\dim L < \infty$) is the **killing form.**

Lemma 3.3: (i) trace forms are invariant symmetric bilinear forms.
(ii) If J is an ideal, then $J^\perp = \{ x : \langle x,y \rangle = 0 \ \forall y \in J\}$ and for an invariant form $\langle \cdot,\cdot \rangle$, then $J^\perp$ is an ideal. In particular, $L^\perp$ is an ideal of L.

proof: Ex: (1) use that trace is invariant: $tr([a,b],c) = tr(a,[b,c]) \quad \forall a,b,c \in \mathrm{End}(V)$.

(i) $tr([a,b]c) = tr(a[b,c]) \quad \forall a,b,c \in \mathrm{End}(V)$.

$[a,b]c = (ab - ba)c = abc - bac$

and $a[b,c] = a(bc - cb) = abc - acb$

But $tr(ab) = tr(ba)$, $\Rightarrow tr(abc - bac) = tr(abc - acb)$

(ii) if J is an ideal, let $J^\perp = \{x : \langle x,y \rangle = 0 \ \forall y \in J\}$. Let $x \in J^\perp$, $y \in L$. Then since $\langle \cdot,\cdot \rangle$ is invariant, $\langle [x,y], z \rangle = \langle x, [y,z] \rangle = 0$ since $x \in J^\perp$. So $[x,y] \in J^\perp \Rightarrow J^\perp$ is an ideal.

Rem: There may be other invariant forms that arent trace forms.

Thm 3.4 (Cartan's criterion for solubility). Let $\mathrm{char}\,K = 0$, and L be a Lie subalgebra of $\mathrm{End}(V)$, $\dim V < \infty$. Let $\langle , \rangle$ be trace form of the embedding $\rho : L \to \mathrm{End}(V)$. Then

$$L \text{ is soluble} \iff \langle x,y \rangle_\rho = 0 \quad \forall x \in L, \ y \in L^{(1)}.$$

$tr(xy) = 0$

Thm 3.5: (cartan - killing criterion for semisimplicity). Let $\mathrm{char}\,K = 0$. Then

$$L \text{ is semisimple} \iff \text{The killing form } \langle \cdot,\cdot \rangle_{ad} \text{ is non-degenerate.}$$

Cartan solubility for repns: $\rho : L \to \mathrm{End}(V)$, $\dim(V) < \infty$, $\mathrm{char}(K) = 0$. Then $\rho(L)$ soluble $\iff \langle x,y \rangle_\rho = 0 \ \forall x \in L, \ y \in L^{(1)}$.

Note: 3.5 is fundamental in the development of the theory of semisimple Lie Algebras.

Lie $\Rightarrow$ (3.4) $\Rightarrow$ (3.5)

Note: 3.5 can be used to show a result about derivations of semisimple Lie Algebras.

Dfn 3.6: A **derivation** of a lie algebra L is a linear map $L \to L$ such that

$$D([x,y]) = [x, Dy] + [Dx, y].$$

Inner derivations are of the form $y \mapsto [x,y]$.

$$\{\text{Inner derivations}\} = ad(L)$$

Thm 3.7 If $\mathrm{char}K = 0$ and $\dim L < \infty$, and L is semisimple, then $\mathrm{Der}(L) = \mathrm{ad}(L)$.

Simple: L has no proper nontrivial ideals

semisimple: $R(L) = 0$, basically $R(L)$ is Σ of all soluble ideals of L. And so $R(L) = 0$ means that L is maximally insoluble in some sense.

Some proofs:

$\boxed{(3.4) \Rightarrow (3.5)}$

$\mathrm{Char}(K) = 0$, $\rho: L \to \mathrm{End}(V)$. L soluble $\Leftrightarrow$ $\langle x,y \rangle_\rho = 0$ $\forall x \in L$, $y \in L^{(1)}$

$\Rightarrow$ L semisimple $\Leftrightarrow$ $\langle \cdot,\cdot \rangle_{\mathrm{ad}(L)}$ nondegenerate.

Proof: L finite dimensional, $\mathrm{char}K = 0$. $R(L)$ radical, $L^\perp$ = orthogonal space wrt killing form
$= (x : \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = 0 \ \forall y \in L)$

Suppose J is an abelian ideal of L. Take $x \in L$, $y \in J$. Then $\mathrm{ad}(y)(L) \subseteq J$ and $\mathrm{ad}(x)\,\mathrm{ad}(y)(L) \subseteq J$ since J is an ideal.

composition of maps.

$[y,x] \in J \ \forall x \in L$

if $x, z \in L$, $y \in J$, then $\mathrm{ad}(y)(z) = [y,z] = j \in J$, and since J is an ideal, $[x,j] \in J$.

abelian:= $\forall x,y \in J$, $[x,y] = 0$

Since J is abelian, $\mathrm{ad}(y)(J) = 0$. Hence, $(\mathrm{ad}(x)\,\mathrm{ad}(y))^2 = \mathrm{ad}(x)\,\mathrm{ad}(y)\,\mathrm{ad}(x)\,\mathrm{ad}(y) = 0$. Hence, $\mathrm{ad}(x)\,\mathrm{ad}(y)$ is a nilpotent endomorphism of L, and so has $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$.

remember we're dealing with symmetric billnear forms

Thus, $\langle x,y \rangle_{\mathrm{ad}} = 0$ $\forall x \in L$, $y \in J$. So, $J \subseteq L^\perp$. But if $R(L) \neq 0$, it contains a nonzero abelian ideal $J \subseteq L$ ($R(L)$ is soluble, take last nonzero term in derived series of $R(L)$). So if $R(L) \neq 0$, then $L^\perp \neq 0$.

$L^\perp \neq 0$ is equivalent to saying that $\langle \cdot,\cdot \rangle$ degenerates somewhere

$\Rightarrow R(L) = 0$.

We proved $\langle , \rangle_{\mathrm{ad}}$ non-degenerate, then L is semisimple.

The converse is a bit more complicated. Suppose L is semisimple, and set $J = L^\perp$, an ideal of L. Consider $\mathrm{ad}_L : L \to \mathrm{end}(L)$, and the image $\mathrm{ad}_L(J)$. We have (by assumption), $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ $\forall x \in J$, $y \in L$ since $J = L^\perp$. In particular, $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$, $\forall x \in J$, $y \in J^{(1)}$. (silly but $J^{(1)} \subseteq L$)

Cartan's solubility criterion (3.4) $\Rightarrow$ $\mathrm{ad}_L(J)$ is soluble. Note that $(\mathrm{ad}(J))^{(1)} = \mathrm{ad}(J^{(1)})$ since ad is a lie algebra homomorphism. But $\ker \mathrm{ad}_L = Z(L)$ = centre of L (commutes with everything), which is an abelian ideal. But by assumption, $R(L) = 0 \Rightarrow \ker \mathrm{ad}_L = 0$ since $Z(L) \subseteq R(L)$. So $J \cong \mathrm{ad}_L(J)$ since ad_L is $\therefore$ injective. I.e. J is soluble. Hence, $J \leq R(L) = 0$, and thus $L^\perp = 0 \Rightarrow \langle , \rangle_{\mathrm{ad}}$ is nondegenerate. $\square$

My version of proof

Suppose L is fin. dim, $\mathrm{Char}K = 0$. Denote $R(L)$ = radical, and $L^\perp$ the orthogonal space to L wrt to killing form. Remark that $\langle \cdot,\cdot \rangle$ is nondegenerate $\Leftrightarrow$ $L^\perp = \{0\}$:

$$L^\perp = \{ x \in L : \langle x \in L : \langle x,y \rangle_{\mathrm{ad}} = 0 \ \forall y \in L \}$$
$$= \{ x \in L : \mathrm{tr}(\underbrace{\mathrm{ad}(x)\,\mathrm{ad}(y)}_{\text{composition}}) = 0 \ \forall y \in L \}$$

Suppose $\langle \cdot,\cdot \rangle_{\mathrm{ad}}$ is nondegenerate. We want to show that L is semisimple ($R(L) = 0$). We'll prove the contrapositive: if $R(L) \neq 0$, then $L^\perp \neq 0$. So suppose instead that $R(L) \neq 0$. Since $R(L)$ is soluble (by dfn its the maximal soluble ideal) then it contains a nonzero abelian ideal $J \subseteq L$:

$R(L)$ soluble $\Rightarrow$ $\exists n \in \mathbb{N}$ s.t $(R(L))^{(n)} = 0$, n minimal. Hence, $[R(L)^{(n-1)}, R(L)^{(n-1)}] = (R(L)^{(n)}) = 0$
$\Rightarrow$ $R(L)^{(n-1)} \leq L$ is abelian (and obvs. an ideal of L)

by induction + Jacobi we see that $L^{(i)}$ is an ideal of L $\forall i$.

So L contains a nonzero abelian ideal by minimality of n. However, consider the following:

Let $x \in L$, $y \in J$. Then $ad(y)(z) = [z,y] \in J$. So $ad(y)(L) \subseteq J$, and again $ad(x)\,ad(y)(L) \subseteq J$. Since J is abelian, $ad(y)(J) = 0$. So $(ad(x)\,ad(y))^2$ is such that for any $z \in L$,

L semisimple $\Leftrightarrow$ $R(L) = 0$.
$R(L) \neq 0$, then $\exists\, n \in \mathbb{Z}$ s.t $(R(L))^{(n)} = 0$, but $(R(L))^{(n-1)} \neq 0$,
Let $J := (R(L))^{(n-1)}$. Then J is abelian, and in particular,
$\langle x,y\rangle_{ad} = 0 \;\forall x \in L,\, y \in J$
$\Rightarrow \langle\cdot,\cdot\rangle_{ad}$ degenerates on J.

$$(ad(x)\,ad(y))^2(z) = \underbrace{ad(x)\,\underbrace{ad(y)\,\underbrace{ad(x)\,ad(y)(z)}_{\in J}}_{=0}}_{=0} \quad \Rightarrow (ad(x)\,ad(y))^2 = 0$$

$\Rightarrow$ $ad(x)\,ad(y)$ is nilpotent endo of L, and so has trace $= 0$

semisimple $\Rightarrow \langle\cdot,\cdot\rangle_{ad}$ nondegenerate.
look at $L^\perp =: J$, and show that, assuming semisimplicity, $J \leq R(L) = 0$.
Let $J = L^\perp$. Then want to show that J is soluble. But since L is semisimple,
$\Leftrightarrow ad(L)$ is semisimple
now $J = L^\perp$, and so $\Rightarrow \forall y \in L, x \in J$,
$\langle x,y\rangle_{ad} = 0$
$\Rightarrow \forall\, y \in J^{(1)}, x \in J$, $\langle x,y\rangle_{ad} = 0$
$\Rightarrow$ by Cartan solubility condition,
$\Rightarrow ad(L) \leq R(L)$.

Hence, $\langle x,y\rangle_{ad} = 0 \;\; \forall\, x \in L, y \in J \Rightarrow 0 \neq J \subseteq L^\perp$. So $L^\perp \neq \{0\} \Rightarrow \langle\cdot,\cdot\rangle_{ad}$ is degenerate.

This proves that $\langle\cdot,\cdot\rangle_{ad}$ nondegenerate $\Rightarrow$ L semisimple.

Now let's prove L semisimple $\Rightarrow \langle\cdot,\cdot\rangle_{ad}$ nondegenerate. Let $J = L^\perp$. We want to show that $J = 0$ if $R(L) = 0$. We can do that if we show $J \leq R(L)$.

Consider the map $ad_L : L \to End(L)$, in particular the image $ad_L(J)$. Since $J = L^\perp$, by assumption $\forall x \in J,\ y \in L$ $\quad tr(ad(x)\,ad(y)) = 0$.

$\langle x,y\rangle_{ad} = 0 \;\; \forall\, x \in J,\ y \in J^{(1)} \subseteq L$

Cartan's criterion for solubility says then that $ad_L(J)$ is soluble. Note that $(ad_L(J))^{(i)} = ad_L(J^{(i)})$ since ad is a Lie algebra homomorphism. But $\ker ad_L = Z(L)$ which is an abelian ideal. The centre is then soluble, so $Z(L) \leq R(L)$. But by assumption $R(L) = 0$, so $\ker ad_L = Z(L) = 0$. Therefore ad_L is injective $\Rightarrow J \cong ad_L(J)$ (iso to its image under ad_L). Since $ad_L(J)$ is soluble, then so is J. Hence $J \subseteq R(L)$. But $R(L) = 0 \Rightarrow J = 0 \Rightarrow L^\perp = 0 \Rightarrow \langle\cdot,\cdot\rangle_{ad}$ is nondegenerate.

$(ad_L(J))^{(i)} = 0 \Leftrightarrow ad_L(J^{(i)}) = 0 \Rightarrow J^{(i)} \subset \ker ad_L = Z(L) \subseteq R(L) = 0 \Rightarrow J^{(i)} = 0 \Rightarrow J$ is soluble $\Rightarrow J \subseteq R(L) = 0$. $\square$

Fact: In general, for finite dimensional L, $[R(L), R(L)] \leq L^\perp \leq R(L)$, but $L^\perp$ and $R(L)$ need not be the same. we showed that $L^\perp \leq R(L)$, but what about $[R(L), R(L)]$? well, say $[x,y] \in L$, $x,y \in R(L)$, and $z \in L$. T

$\boxed{\text{Lie / Engel} \Rightarrow (3.4)}$ for algebraically closed k, char $k = 0$.

proof: WTS: $L \leq End(V)$ soluble $\Rightarrow tr(xy) = 0 \;\; \forall\, x \in L,\ y \in L^{(1)}$. But this follows quickly from the corollary of Lie's Thm:

L soluble $\Rightarrow$ $\exists$ basis of V wrt which L is represented by upper triangular matrices: $L \subseteq \mathfrak{b}_n$

$\Rightarrow L^{(1)} \subseteq \mathfrak{n}_n$ = strictly upper triangular matrices.

If $x \in L$, $y \in L^{(1)}$, then xy has zero entries on leading diagonal $\Rightarrow tr(xy) = 0 \;\; \forall\, x \in L, y \in L^{(1)}$.

Argument for converse is much more complicated. Assuming the trace condition, we want to show L is soluble. For that its enough to show that $L^{(1)}$ is nilpotent. We want to apply Engel (2.19), and so we need to establish that elements of $L^{(1)}$ are nilpotent endomorphisms. We'll need some preparatory linear algebra.

Dfn 3.8 : $x \in \mathrm{End}(V)$ is semisimple $\Leftrightarrow$ it is diagonalizable.
$\Leftrightarrow$ minimal polynomial is a product of distinct linear factors.

Rem 1) if x is semisimple, $x(W) \le W$ for a subspace $W \le V$, then $x|_W : W \to W$ is semisimple
2) if x, y semisimple and $xy = yx$, then x, y are simultaneously diagonalizable, and $x \pm y$ is also semisimple.

Lemma 3.9 (Jordan decomposition)

For $x \in \mathrm{End}(V)$,

(1) $\exists$ unique $x_s, x_n \in \mathrm{End}(V)$ with x_s semisimple, x_n nilpotent, and x_n, x_s commute, and $x = x_s + x_n$.

(2) $\exists$ polynomials $p(t), q(t)$ with zero constant term such that $x_s = p(x)$, and $x_n = q(x)$. So, x_s, x_n commute with all endomorphisms that commute with x.

(3) If $U \le W \le V$ and $x(W) \subseteq U$, then $x_s(W) \subseteq U$ and $x_n(W) \subseteq U$.

Dfn 3.10 : x_s, x_n are called the semisimple and nilpotent parts of x respectively.

Exm : if x is represented by $\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ (Jordan normal form). Then $x_s = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$, $x_n = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

Over an algebraically closed field, we know that x can be represented by its Jordan normal form, which we can split in a similar fashion. It is the uniqueness that is harder to prove.

pf of 3.9: (ii) $\Rightarrow$ (iii) immediately. $x_s = p(x)$, if $x(W) \subseteq U$, then $p(x)(W) \subseteq U \Rightarrow x_s(W) \subseteq U$, similarly for x_n. (a poly in x) no constant term.

pf of (ii) Let $\prod (t - \lambda_i)^{m_i}$ be the characteristic polynomial of x, and $V_i = \ker(x - \lambda_i \iota)^{m_i}$ for each i (i.e the generalized λ_i - eigenspace) $V = \oplus V_i$. partitions V.

Then the characteristic polynomial of $x|_{V_i}$ is $(t - \lambda_i)^{m_i}$. Find a polynomial such that $p(t) \equiv 0 \mod t$, $p(t) \equiv \lambda_i \mod (t - \lambda_i)^{m_i}$. This exists by Chinese remainder Theorem.

Define $q(t) = t - p(t)$. Set $x_s = p(x)$, $x_n = q(x)$. Then p and q have zero constant term.
since $p(t) \equiv 0 \mod t$

On V_i, $x_s - \lambda_i \iota$ acts like a multiple of $(x - \lambda_i)^{m_i}$, and so trivially. Thus V_i is an eigenspace for x_s, i.e. x_s is diagonalizable (V_i is the λ_i-eigenspace of x_s). Also note that $x_n = x - x_s$ acts like $x - \lambda_i \iota$ on V_i and hence nilpotently. Thus x_n is nilpotent.
(lo)

Uniqueness of (i) : if $x = s + n$, s semisimple and n nilpotent, then n and s both commute with x, and hence with x_s and x_n. So considering

$$x_s - s = n - x_n$$

both sides are semisimple, and nilpotent, so they must be zero! Hence $x_s = s$, $x_n = n$. So uniqueness holds. $\square$

if $L^{(1)}$ is nilpotent, then $\exists$ a $c \in \mathbb{N}$ s.t $(L^{(1)})_{(c)} = 0 \Rightarrow (L^{(1)})_{(2^c)} = 0$. But from example sheet 1,

$$(L^{(1)})^{(m)} \subseteq (L^{(1)})_{(2m)} \quad \forall m \Rightarrow (L^{(1)})^{(2^c)} = 0$$

$$\Rightarrow L^{(2^c+1)} = 0 \text{ say.}$$

Set $x_s = p(x)$, $x_n = q(x)$. Since they are polynomials in x, x_s and x_n commute with each other, as well as with all endomorphisms which commute with x. They also stabilize all subspaces of V stabilized by x, in particular the V_i. The congruence $p(T) \equiv a_i \pmod{(T-a_i)^{m_i}}$ shows that the restriction of $x_s - a_i \cdot 1$ to V_i is zero for all i, hence that x_s acts diagonally on V_i with single eigenvalue a_i. By definition, $x_n = x - x_s$, which makes it clear that x_n is nilpotent. Because $p(T)$, $q(T)$ have no constant term, (c) is also obvious at this point.

semisimple and nilpotent element $= 0$:

semisimple $\Rightarrow$ diagonalizable $\to$ diagonalizes to matrix with eigenvalues on diagonal
nilpotent $\to$ + $\checkmark$
$\| $ zero matrix.
only eigenvalue is 0

Lemma 3.11: If $x \in L \subseteq End(V)$, let x_s and x_n be the semisimple / nilpotent parts. Then $ad(x_s)$ and $ad(x_n)$ are the semisimple / nilpotent parts of $ad(x)$.

sum of all soluble ideals in L, and $\forall x,y \in Z(L)$, $[x,y]=0$ since abelian $\Rightarrow Z(L) \subseteq R(L)$.

Remark: if L is semisimple (and so $Z(L) \subseteq R(L) = 0$), we know that $L \cong ad(L) \subseteq End(L)$. And we can say that $x \in L$ is semisimple if $ad(x)$ is semisimple.

Pf: First, observe that x_n is nilpotent $\Rightarrow ad(x_n)$ is nilpotent: Suppose $x_n \in L \subseteq End(V)$ with $x_n^m = 0$ for some $m \in \mathbb{Z}$. Define a map $\Phi(x_n): End(V) \to End(V)$; $y \mapsto x_n y$ (with composition), and $\Theta(x_n): End(V) \to End(V)$, $y \mapsto y x_n$. Then $\Phi(x_n)$ and $\Theta(x_n)$ commute, and $ad(x_n)$ is the restriction of $\Phi(x_n) - \Theta(x_n)$ to L.
Since $x_n^m = 0$, we have that $\Phi(x_n^m) = 0 = \Theta(x_n^m)$. Consider then that

$$(ad(x_n))^r = (\Phi(x_n) - \Theta(x_n))^r$$
$$= 0 \quad \text{for } r \geq 2m-1$$

using fact that $\Phi(x_n)$ and $\Theta(x_m)$ commute, and expanding by Binomial theorem.

So $ad(x_n)$ is also nilpotent.

Note that x_s, x_n commute $\Rightarrow ad(x_s)$ and $ad(x_n)$ commute. But ad is a linear map, so $ad(x) = ad(x_s) + ad(x_n)$.
It remains to show that $ad(x_s)$ is semisimple. The fact that x_s is semisimple $\Rightarrow \exists$ basis of eigenvectors in V, $x_s(v_i) = \lambda_i v_i$ say. Define maps $\theta_{ij} \in End(V)$, $v_i \mapsto v_j$, and $v_\ell \mapsto 0$ $\ell \neq i$ corresponding to an elementary matrix. Notice that $x_s \theta_{ij}(v_i) = \lambda_j v_j$, $x_s \theta_{ij}(v_\ell) = 0$. Also note that $\theta_{ij} x_s(v_i) = \lambda_i v_j$, $\theta_{ij} x_s(v_\ell) = 0$.
Thus $ad(x_s)(\theta_{ij}) = (\lambda_j - \lambda_i)\theta_{ij}$. I.e. θ_{ij} form a basis of eigenvectors of $ad(x_s): End(V) \to End(V)$. Thus, we know that $ad(x_s)$ is diagonalizable, and so its restriction to $L \subseteq End(V)$, $ad(x_s)|_L : L \to L$ is diagonalizable.
So $ad(x_s)$ is semisimple.

$x_s(v_i) = \lambda_i v_i$, define $\theta_{ij} \in End(V)$, $v_i \mapsto v_j$, $v_\ell \mapsto 0$ $\forall \ell \neq i$. Then $x_s \theta_{ij}(v_i) = x_s(v_j) = \lambda_j v_j$, and $x_s \theta_{ij}(v_\ell) = x_s(0) = 0$. Then $ad(x_s)(\theta_{ij}) = [x_s, \theta_{ij}] = x_s\theta_{ij} - \theta_{ij}x_s$

acting on any v_k, $= (x_s\theta_{ij} - \theta_{ij}x_s)(v_k)$
$= x_s \theta_{ij}(v_k) - \theta_{ij}(\lambda_k v_k)$

When $k \neq i$, map $\equiv 0$. When $k = i$, $\longrightarrow$ $= \lambda_j \theta_{ij}(v_k) - \lambda_i \theta_{ij}(v_k)$
$= (\lambda_j - \lambda_i)\theta_{ij}$.

$\square$

Lemma 3.12: Let A and B be subspaces of $End(V)$, with $A \subseteq B$, and let $T = \{t \in End(V) : [t, B] \subseteq A\}$. Let $w \in T$ and suppose w satisfies $tr(wt) = 0$ $\forall t \in T$. Then w is nilpotent.

pf: Let $w = w_s + w_n$ semisimple/nilpotent parts. We want to show $w_s = 0$. Pick $v_1, \dots, v_n$ a basis of eigenvectors of w_s, $w_s(v_i) = \lambda_i v_i$. Define θ_{ij} as in the previous proof of (3.11). We have that $ad(w_s)(\theta_{ij}) = (\lambda_j - \lambda_i)\theta_{ij}$ as before. Assume $w_s \neq 0$, then $\exists i$ s.t $\lambda_i \neq 0$. Let $E = \mathbb{Q}$ - span of $\lambda_1, \dots, \lambda_n$, $f: E \to \mathbb{Q}$ a linear form and choose it to be nonzero. Set $y(v_i) := f(\lambda_i) v_i$.
So $ad(y)(\theta_{ij}) = (f(\lambda_j) - f(\lambda_i))\theta_{ij} = (f(\lambda_j - \lambda_i))\theta_{ij}$. by linearity of f. Let $r(t)$ be a polynomial with zero constant term, so that $r(\lambda_j - \lambda_i) = f(\lambda_j - \lambda_i)$ $\forall i,j$. Then $ad(y) = r(ad(w_s))$. By 3.11, $ad(w_s)$ is the semisimple part of $ad(w)$, and is a polynomial in $ad(w)$ with zero constant term by lemma 3.9 (ii).
So $ad(y)$ is also such a polynomial expression.

But $w \in T$ and so $[w, B] \subseteq A$ i.e. $ad(w)(B) \subseteq A$. So $ad(y)(B) \subseteq A$. By supposition $tr(wt) = 0$ $\forall t \in T$. and so $tr(wy) = 0$. But $tr(wy) = \sum_i \lambda_i f(\lambda_i) \in \mathbb{Q}$. But f is linear and so applying f, we get $\sum_i (f(\lambda_i))^2 = 0$. So $f(\lambda_i) = 0$, and hence f has to be the zero form $\lightning$.

So $w_s = 0$, i.e. w is nilpotent.

$\square$

Thm 3.4 (Cartan's Criterion for Solubility). Let char $k = 0$, and L be a Lie subalgebra of $\mathrm{End}(V)$, $\dim V < \infty$. Let $\langle , \rangle$ be trace form of the embedding $\rho : L \to \mathrm{End}(V)$. Then

(*) L is soluble $\Leftrightarrow$ $\langle x, y \rangle_\rho = 0$ $\forall x \in L$, $y \in L^{(1)}$.

Now back to the proof of the Cartan Solubility Criterion (3.4)!

We're trying to show that the trace condition implies solubility. We'd observed that it was enough to show that the derived subalgebra $L^{(1)}$ consisted of nilpotent endomorphisms. Suppose RHS of * holds.

Take $A = L^{(1)}$ and $B = L$ in 3.12. So $T = \{ t \in \mathrm{End}(V) : [t, L] \leq L^{(1)} \}$. Notice that $L^{(1)} \subseteq T$ as $L^{(1)}$ is an ideal of L. Recall $L^{(1)}$ is spanned by $[x, z]$, $x, z \in L$. Let $t \in T$.
But $\mathrm{tr}([x,z]t) = \mathrm{tr}(\underset{L}{x}\underset{L^{(1)}}{[z,t]}) = 0$ by assumption $(\langle x, y \rangle_\rho = 0 \ \forall x \in L, \ y \in L^{(1)})$

So $\mathrm{tr}(wt) = 0$ $\forall w \in L^{(1)}$, and $t \in T$. But $L^{(1)} \leq T$ and so w is nilpotent $\forall w \in L^{(1)}$ by 3.12. $\square$

Proof of Engel's Theorem:

recall: Thm 2.19 : (Engel) Suppose $L \leq \mathrm{End}(V)$ is a Lie subalgebra, $\dim V < \infty$, and every element of L is a nilpotent endomorphism (i.e. $\forall x \in L$, $\exists a \in \mathbb{N}$ s.t $x^a = 0$). Then $\exists$ $v \neq 0$, $v \in V$ such that $x(v) = 0$ $\forall x \in L$.

Proof by induction on $\dim L$.

Clearly true when $L = 0$. If $\dim L = 1$, then $L = \langle x \rangle$. Then x nilpotent (eigenvalues are zero) $\Rightarrow x(v) = 0$ for some $v \neq 0$.

Suppose $\dim V \geq 2$ and assume result holds for smaller dimensions. Let L_1 be a maximal (wrt. proper) Subalgebra of L. Note that $\dim L_1 \geq 1$ since $\langle x \rangle$ is a Lie subalgebra for any $x \in L$. Since L_1 is a (Lie) Subalgebra, we can define $\pi : L_1 \longrightarrow \mathrm{End}(L/L_1)$; $x \longmapsto (y + L_1 \longmapsto [x,y] + L_1)$.
Note that $\dim \pi(L_1) \leq \dim L_1 \lneq \dim L$. Moreover, $\pi(L_1)$ consists of nilpotent endomorphisms (similar argument to one used at the beginning of 3.11). Applying the inductive hypothesis, $\exists$ $y \in L/L_1$ $(\notin L_1)$ such that $\pi(x)(y + L_1) = 0$ $\forall x \in L_1$. This implies $[x,y] \in L_1$ $\forall$ $x \in L_1$ (*).

Note that $y \notin L_1$. So $L_1 + \langle y \rangle$ is a lie subalgebra of L and strictly contains L_1. But by maximality of L_1, $L_1 + \langle y \rangle = L$. Also, this shows L_1 is an ideal of L.

Using induction again (this time on Lie ideal (subalgebra) L_1), $\exists$ $v \neq 0 \in V$ such that $x(v) = 0$ $\forall x \in L_1$. Let $V_0 = \{ v \in V : x(v) = 0 \ \forall x \in L_1 \} \neq 0$. Then $y(V_0) \subseteq V_0$: to see this, note:

$$
\begin{aligned}
x(y(v)) &= ([x,y] + yx)(v) \\
&= [x,y](v) + y(x(v)) && [x,y] \in L_1 \text{ by } (*) \\
&= 0 + y(0) = 0. && \text{and } x(v) = 0 \ \forall x \in L_1.
\end{aligned}
$$

And so, $x(y(v)) \in V_0$ for all $x \in L_1$. Therefore V_0 contains a $0 \neq v_0$ with $y(v_0) = 0$ since $y|_{V_0}$ is nilpotent.
Thus, $L(v_0) = 0$. So we're done.

Rem: There's no restriction on the field. $\square$

> Basic idea: we know $L = L_1 + \langle y \rangle$, where L_1 is some maximal proper subalgebra of L, and y is some endomorphism given by the above argument. Now, by induction $\dim(L_1) < \dim(L) \Rightarrow \exists$ $v \neq 0$ s.t $\forall x \in L_1$, $x(v) = 0$. We then look at <u>all</u> the possible nonzero v that satisfy this, $V_0 = \{ v \neq 0 \in V : x(v) = 0 \ \forall x \in L_1 \}$, and then show that $\exists$ $v \in V_0$ s.t $y(v) = 0$ too. Since $L = L_1 + \langle y \rangle$, $\Rightarrow x(v) = 0$ $\forall x \in L$, proving the claim.

Proof of Lie's Theorem (2.18)

Recall: Thm 2.18 : (Lie) For algebraically closed k, char $k = 0$. Suppose $L \subseteq \mathrm{End}(V)$ with $\dim V < \infty$. Suppose L is soluble. Then $\exists\, v \in V$, $v \neq 0$, such that $x(v) = \lambda v$ for all $x \in L$.

Rem: dealing exclusively with algebraically closed field, of char $k = 0$.

Proof by inducting on $\dim L$: Clearly holds for $L = 0$ ✓

otherwise we'd get stuck in a loop of $[L, L]$

$[x,y] = 0 \;\; \forall\, x, y \in L/L^{(1)}$.

Assume $\dim L > 0$. Then L soluble $\Rightarrow L^{(1)} \subsetneq L$. Note that $L/L^{(1)}$ is an Abelian Lie algebra, so any subspace of $L/L^{(1)}$ is an ideal of $L/L^{(1)}$. Take $L^{(1)} \subseteq L_1 \subsetneq L$, so that $\dim(L/L_1) = 1$. Note that $L_1/L^{(1)}$ is a subspace of $L/L^{(1)}$, therefore an ideal, and hence, L_1 is an ideal in L.

> $L_1/L^{(1)}$ is a subspace of $L/L^{(1)}$, and $\therefore$ an ideal. Hence, L_1 is an ideal in L:
> Note that if $x \in L_1$, and $y \in L$, then $[x,y] \in L_1$ since
>
> $$[(x + L^{(1)}), (y + L^{(1)})] = [x,y] + L^{(1)}$$
>
> and $L/L^{(1)}$ is abelian $\Rightarrow [x,y] + L^{(1)} = 0 + L^{(1)} \Rightarrow [x,y] \in L^{(1)} \subseteq L_1 \Rightarrow [x,y] \in L_1$.

Use induction to see that L_1 has a common eigenvector, $x(v) = \lambda_x v \;\; \forall\, x \in L_1$. The map $x \mapsto \lambda_x$, $L_1 \to k$ is a linear form.

Let $W = \{ w \in V : x(w) = \lambda_x w \;\; \forall\, x \in L_1 \} \neq 0$ ← since $v \in W$. Think of W as a "common eigenspace".

L_1 is of codimension 1, so $L = L_1 + \langle y \rangle$ for some $y \in L$. We'll show that $L(W) \subseteq W$:

Certainly $L_1(W) = W$ by construction, so we just need to confirm this for y : $y(W) \subseteq W$. But

$$\begin{aligned} x(y(w)) &= (yx + [x,y])(w) \\ &= yx(w) + [x,y](w) \\ &= y(\lambda_x w) + [x,y](w) \\ &= \lambda_x y(w) + \lambda_{[x,y]} w \end{aligned}$$

since $[x,y] \in L_1$ (L_1 an ideal)

We'll get what we want if we can show $\lambda_{[x,y]} = 0$. Then $y(W) \subseteq W$.

Take some $w \in W$, $w \neq 0$, and let $U_n = \langle w, y(w), \dots, y^{n-1}(w) \rangle$. Then $\langle w \rangle = U_1 \subsetneq U_2 \subsetneq \dots$ must terminate at some U_r, but up to that point, U_n has basis $w, y(w), \dots, y^{n-1}(w)$ (linear independence).

We'll show that L_1 leaves each U_n invariant (if $L_1(U_n) \subseteq U_n$).

Now, $U_1 = \langle w \rangle$, and $x(w) = \lambda_x w \in U_1 \;\; \forall\, x \in L_1$ so the beginning is obvious. For U_2, $U_2 = \langle w, y(w) \rangle$. We saw

$$\begin{aligned} x(y(w)) &= y(x(w)) + [x,y](w) \\ &= \lambda_x y(w) + \lambda_{[x,y]} w \in U_2. \end{aligned}$$

Continuing onwards, we get that on U_n, using the given basis, x is represented by an upper triangular matrix

$$\begin{pmatrix} \lambda_x & \lambda_{[x,y]} & \cdots \\ & \ddots & 0 \\ 0 & & \lambda_x \end{pmatrix}$$

across top you get
$\lambda_x, \lambda_{[x,y]}, \lambda_{[[x,y],y]}, \dots$

Thus, $x(U_n) \subseteq U_n$ for each $x \in L_1$, and $x|_{U_n}$ is represented by a matrix of trace $n\lambda_x$.
Observe that U_r is invariant under y: $y(U_r) \subseteq U_r$. Thus, U_r is invariant under $L = L_1 + \langle y \rangle$.

Notice that $[x,y]|_{U_r}$ is represented by a matrix of trace $r \cdot \lambda_{[x,y]}$ (because $[x,y] \in L_1$).
But $[x,y]|_{U_r}$ must have trace zero, since commutators of endomorphisms have trace zero.
But $\operatorname{char} K = 0 \Rightarrow \lambda_{[x,y]} = 0$, as needed to complete the proof:

W is invariant under L. Because K is algebraically closed, $\exists\, w \in W$, $w \neq 0$ an eigenvector for y. This w is a common eigenvector for all of L. $\square$

Finally for this chapter:

Proposition 3.13: Let L be a finite dimensional Lie algebra, $\operatorname{char} K = 0$.
(1) if L is semisimple, then L is a direct sum of non abelian, simple ideals.
(2) if $0 \neq J$ is an ideal of $L = \oplus L_i$, then the ideal is a direct sum of some of the L_i
(3) if L is a direct sum of non abelian simple ideals, then L is semisimple.

Proof: (1) induction on $\dim L$
Let J be an ideal in semisimple L. By 3.5, the killing form on L is nondegenerate. The orthogonal space $J^\perp$ is an ideal (Q10). In particular,

$$\dim J + \dim J^\perp = \dim L$$

But $J \cap J^\perp$ is soluble and an ideal (by Cartan Solubility criterion, 3.4, applied to $\operatorname{ad}(J \cap J^\perp)$), and so is zero, since L is semisimple. Hence $L = J \oplus J^\perp$. Note that any ideal of J is an ideal of L, and similarly for $J^\perp$. So J and $J^\perp$ are both semisimple.

> J is semisimple, and $J^\perp := \{ x \in L : \langle x, y \rangle_{ad} = 0 \ \forall\, y \in J \}$, the orthogonal space wrt. the killing form. Cartan's solubility criterion says that $K := J \cap J'$ is soluble iff $\langle x, y \rangle_{ad} = 0 \ \forall\, x \in K,\ y \in K^{(1)}$. But (1) $K^{(1)} \subseteq K$ since K is an ideal (simple calc), and 2, $\forall\, x \in K,\ y \in K^{(1)}$, $x \in J$ and $y \in J^\perp$, $\Rightarrow$ by dfn of $J^\perp$ $\langle x, y \rangle_{ad} = 0$. Hence K is soluble. Since $J \cap J^\perp$ is a soluble ideal of L, $\Rightarrow J \cap J^\perp \subseteq R(L)$.
> But by assumption L is semisimple, so $J \cap J^\perp \subseteq R(L) = 0 \Rightarrow J \cap J^\perp = 0 \Rightarrow L = J \oplus J^\perp$.
>
> Also, any ideal $M \subseteq J$ is an ideal of L: this is true because of the splitting. For $x \in M \subseteq J$, then $\forall\, z \in L$, we can write $z = z_0 + z_1$, where $z_0 \in J$ and $z_1 \in J^\perp$. Then
>
> $$[x, z] = [x, z_0] + [x, z_1]$$
>
> Now, $z_0 \in J$ and $x \in M \Rightarrow [x, z_0] \in M$. And $x \in J$, $z_1 \in J^\perp$, and $J, J^\perp$ both ideals means that $[x, z_1] \in J \cap J^\perp = 0 \Rightarrow [x, z_1] = 0$. Thus $[x, z] \in M + 0 = M \ \forall\, x \in M, z \in L \Rightarrow M$ is an ideal in L.
>
> Hence $R(J), R(J^\perp) \subseteq R(L)$ (sum of all soluble ideals in L, and if say $K \subseteq R(J)$ is soluble in J, then it is soluble in L too). So L semisimple $\Rightarrow J$ and $J^\perp$ semisimple too.

Synopsis: induct on $\dim L$.
1) if L not simple, $\exists\, 0 \neq J \trianglelefteq L$ ideal
2) By $\langle \cdot, \cdot \rangle_{ad}$ nondegenerate, $J \oplus J^\perp = L$.
3) L semisimple $\Rightarrow J$ semisimple
4) if $M \trianglelefteq J \trianglelefteq L$, then $M \trianglelefteq L$.

By induction, J and $J^\perp$ are direct sums, as desired.

(ii) If $J \cap L_i = 0$, then $[L_i, J] = 0$ since L_i, J are ideals, and hence $J \subseteq \bigoplus_{j \neq i} L_j$ (we're using that L_i has zero centre). If $J \cap L_i \neq 0$, then the simplicity of $L_i \Rightarrow J \cap L_i = L_i \Rightarrow L_i \leq J$.
Hence $J = \bigoplus_{L_j \leq J} L_j$.

(iii) If L is a direct sum of non abelian simple ideals, then by (ii), $R(L)$ will be a direct sum of some of the L_i. But $R(L)$ is soluble, and so cannot contain nonabelian, simple ideals. So $R(L) = 0$, and hence L is semisimple.

> Suppose J is a nonabelian, simple ideal of K. Then J only has ideals 0 and J. Also, nonabelianness implies that $\exists\, x \in J$ s.t $[x, J] \neq 0$. Hence since $[x, J] \subseteq [J, J] = J^{(1)}$, so $J^{(1)} \neq 0$ and is an ideal $\Rightarrow J^{(1)} = J$, which is $\therefore$ not soluble. But $R(L)$ is soluble, so it must be that $R(L) = 0$.

4 CARTAN SUBALGEBRAS AND WEIGHT DECOMPOSITION

Throughout, L is finite dimensional over $\mathbb{C}$.

Dfn 4.1: $L_{\lambda, y} = \{ x \in L : (\mathrm{ad}(y) - \lambda \underbrace{i}_{\text{identity map}})^r (x) = 0 \}$ is the generalized λ-eigenspace for $\mathrm{ad}(y)$ $(y \neq 0)$.

Note: $y \in L_{0,y}$ since $[y, y] = 0$. We write $L_{\lambda, y} = 0$ if λ is not actually an eigenvalue of $\mathrm{ad}(y)$.

Note: $L = \bigoplus L_{\lambda, y}$ is a direct sum of generalized λ-eigenspaces. (By Primary Decomposition Theorem)
sum over all λ's.

Lemma 4.2:
(i) $[L_{\lambda,y}, L_{\mu,y}] \subseteq L_{\lambda+\mu, y}$
(ii) $L_{0,y}$ is a Lie subalgebra

pf: (ii) is immediate from 1. — $[L_{0,y}, L_{0,y}] \subseteq L_{0+0,y} = L_{0,y}$

☆ derivation / $[\cdot,\cdot]$ property
★ $\mathbb{C}$ - linearity.

(i) Consider
$$(\mathrm{ad}(y) - (\lambda+\mu)i)[x, z]$$
$$= [(\mathrm{ad}(y) - \lambda i)x, z] + [x, (\mathrm{ad}(y) - \mu i)z]$$

$= \mathrm{ad}(y)([x,z]) - (\lambda+\mu)i([x,z])$
$= [\mathrm{ad}(y)(x), z] + [x, \mathrm{ad}(y)(z)] - \lambda[x,z] - \mu[x,z]$
$= [\mathrm{ad}(y)(x), z] + [x, \mathrm{ad}(y)(z)] + [-\lambda x, z] + [x, -\mu z]$
$= [(\mathrm{ad}(y) - \lambda)(x), z] + [x, (\mathrm{ad}(y) - \mu)(z)]$.

and so
$$(\mathrm{ad}(y) - (\lambda+\mu)i)^n [x, z]$$
$$= \sum [(\mathrm{ad}(y) - \lambda i)^i(x), (\mathrm{ad}(y) - \mu i)^j z]$$

for sufficiently large n these terms all vanish.

$i+j=n$ so $\Sigma = 0$.

Hence, if $x \in L_{\lambda,y}$, $z \in L_{\mu,y}$, then $[x,z] \in L_{\lambda+\mu,y}$.

Dfn 4.3: A Cartan subalgebra (CSA) H of L is nilpotent ($L_{(c)}=0$ for some $c\in\mathbb{Z}$) and self idealising: $\underbrace{\{x : [x,H] \subseteq H\}}^{\text{idealiser}} = H$

Theorem 4.4 (Cartan) [Existence of CSAs]

H is a Cartan Subalgebra $\Leftrightarrow$ H is a minimal Subalgebra of the form $L_{0,y}$.
All CSAs have the same dimension

Thm 4.6 (not proved here)
Any two CSAs are conjugate under the group of automorphisms of L, which are generated by

$$e^{ad(y)} = 1 + ad(y) + \frac{(ad(y))^2}{2!} + \dots$$ with ad(y) nilpotent (i.e. finite sum).

Thm 4.7 (not proved here)

The set of regular elements (elements $y \in L$ s.t $L_{0,y}$ is a CSA) is connected.
Ie. Zariski dense, open subset of L

Example: $L = sl_2$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$.

Then $L_{0,h} = \langle h \rangle$, $L_{2,h} = \langle e \rangle$, $L_{-2,h} = \langle f \rangle$. And $L = sl_2 = L_{0,h} + L_{2,h} + L_{-2,h}$, and $[L_{2,h}, L_{-2,h}] \subseteq L_{0,h}$. Further, $L_{0,h} = \langle h \rangle$ is a CSA, clearly can't have anything smaller (its minimal).

Note: $L_{0,y}$ is always nonzero since $y \in L_{0,y}$ in general].

Theorem 4.8: Let H be a CSA of a semisimple L. Then
a) it is a maximal abelian subalgebra
b) every element of H is semisimple.
c) The restriction of the killing form $\langle\cdot,\cdot\rangle_{ad}$ of L to H is also nondegenerate.

Proof: c) $H = L_{0,y}$ for some (regular) y by 4.4. Consider the decomposition $L = L_{0,y} \oplus \left(\sum_{\lambda \neq 0} L_{\lambda,y}\right)$

By 4.2, $\Rightarrow [L_{\lambda,y}, L_{\mu,y}] \subseteq L_{\lambda+\mu,y}$. So take $u \in L_{\lambda,y}$, $v \in L_{\mu,y}$, with $\lambda+\mu \neq 0$. Then applying $ad(u)ad(v)$, this maps each generalized eigenspace to a different one.

3 So $tr(ad(u)ad(v)) = 0$. Thus when $\mu+\lambda \neq 0$, $L_{\lambda,y}$ is orthogonal to $L_{\mu,y}$ wrt. the Killing form $\langle\cdot,\cdot\rangle_{ad}$

So $L = L_{0,y} \oplus (L_{\lambda,y} + L_{-\lambda,y}) \oplus$ (the rest of the guys) is an orthogonal direct sum.

But Cartan-Killing (3.5) $\Rightarrow$ $\langle\cdot,\cdot\rangle_{ad}$ is nondegenerate. (we assumed L is semisimple)

So its restriction to each direct summand is non degenerate, i.e. the restriction to $L_{0,y}$ is nondegenerate.

nilpotent $\Rightarrow$ soluble $\qquad \langle H, H^{(1)} \rangle_{ad} = 0$

ⓐ H nilpotent (from dfn of CSAs), Cartan solubility (3.4) $\Rightarrow$ $H^{(1)}$ is orthogonal to H wrt $\langle \cdot,\cdot \rangle_{ad}$.
But we've just shown that the restriction to H of $\langle \cdot,\cdot \rangle_{ad}$ is nondegenerate. Hence $H^{(1)} = 0$.
That is, H is abelian. $[H,H]=0$.

$L_{0,y} = \{ x \in L : (ad(y))^r = 0 \} \supseteq \{ x \in L : (ad(y))^1 = 0 \}$

To see maximality: $H = L_{0,y}$ for some $y \in L$. Then $H = L_{0,y} \supseteq \{ z \in L : [y,z] = 0 \} \supseteq H$ since H is abelian.
But if $H_1 \geq H$ is abelian, then H_1 commutes with y (since $y \in H$) and so $H_1 = H$.
$\Rightarrow H_1 \subseteq \{ z \in L : [y,z] = 0 \} \subseteq H \Rightarrow H_1 = H$.

ⓑ Take $x \in H$. Let $x = x_s + x_n$ be the Jordan decomposition of x. If h commutes with x, then h commutes with x_s and x_n ($ad\, x$ is injective).

Recall that $ad_x = ad\, x_n + ad_{x_s}$ nilpotent / semisimple components. $\qquad$ Semisimple = diagonalizable.

We know that H is abelian, and so commutes with x $\forall x \in H$, and hence H commutes with x_n too.
But if $x_n \notin H$, then $H + \langle x_n \rangle$ is an abelian subalgebra of L larger than H, a contradiction by the maximality of H. So $x_n \in H$.

$$\begin{aligned} ad\, x_n \text{ nilpotent} &\Rightarrow ad(h)\, ad(x_n) \text{ is nilpotent (using commutativity)} \\ &\Rightarrow tr(ad(h)\, ad(x_n)) = 0 \qquad \text{nilpotent maps have trace} = 0. \\ &\Rightarrow \langle h, x_n \rangle_{ad} = 0 \quad \forall h \in H. \end{aligned}$$

But $x_n \in H$ and we've shown that the restriction $\langle \cdot,\cdot \rangle_{ad}$ to H is nondegenerate,
So it must be that $x_n = 0$. Hence $x = x_s \Rightarrow x$ is semisimple. $\square$

Lemma 4.9: (converse of 4.8). Let H be a maximal abelian subalgebra $\subseteq L$, all of whose elements are semisimple. Then H is a Cartan subalgebra.

pf: H is abelian $\Rightarrow$ H is nilpotent. All left to show is self-idealising: i.e. $\{ x \in H : [x,H] \subseteq H \} = H$.

If $[x,H] \subseteq H$, then $x \in L_{0,y}$ $\forall y \in H$. But y is semisimple, and so $L_{0,y}$ is diagonalisable.
I.e. $L_{0,y}$ is the 0-eigenspace for $ad(y)$. (not just the generalized eigenspace)

Since H is abelian, if $[x,H] \subseteq H$, then $\forall y \in H$, $[x,y] \in H \Rightarrow [y,[x,y]] = 0 \Rightarrow x \in L_{0,y}$.

So x commutes with y $\forall y \in H$. And so $H + \langle x \rangle$ is an abelian subalgebra. Maximality $\Rightarrow$ $x \in H$.
Thus H is self-idealising $\square$

Remark: Some authors when just talking about semisimple, define CSAs as maximal abelian subalgebras, all of whose elements are semisimple.

Corollary 4.10 (of 4.8): **Regular elements of semisimple L are semisimple.**

pf: y regular $\Rightarrow$ $L_{0,y}$ CSA
But $y \in L_{0,y}$ $\Rightarrow$ y is semisimple by 4.8.

Now suppose L is a semisimple complex Lie Algebra. Take H to be a CSA, with basis $h_1, \dots, h_n$. An easy induction on the dimension of H shows that L decomposes as the direct sum of common eigenspaces for $ad(H)$. need to think of proof.

(using that $ad(H)$ is abelian, and elements are diagonalisable). Each such common eigenspace is of the form L_α

$$L_\alpha := \{ x \in L : ad(h)(x) = \alpha(h)(x) \ \forall h \in H \}$$

$\alpha : H \to \mathbb{C}$ is a linear form.

rather than fixing an h and letting x run through, you sort of fix x and let h run through, the eigenvalues dependent on this.

idea: specify a linear form α

Notice that $L_0 = H$ since H is maximal abelian $L_0 = \{ x \in L : ad(h)(x) = 0 \ \forall h \in H \} = \{ x \in L : [h, x] = 0 \}$

abelian $\Rightarrow H \subseteq L_0$, but H maximal then $\Rightarrow =$.

also called "root space decomposition".

Dfn 4.11: The Weight Space or Cartan decomposition of semisimple L wrt. CSA' H.

$$L = L_0 \oplus \Big(\bigoplus_{\alpha \neq 0} L_\alpha \Big) \quad \text{with} \quad L_0 = H$$

The nonzero elements of L_α have weight α

The $L_\alpha \neq 0$ are the weight spaces (Sometimes useful to write L_α even if its zero).

The nonzero weights are called the roots of L (wrt H).

Notation: Φ = set of roots.
$m_\alpha = \dim L_\alpha$
$\langle \cdot, \cdot \rangle$ = killing form

talking strictly about complex Lie Algebras!

Remark: the following analysis depends on this decomposition. However, even in the real case there are semisimple real Lie algebras that don't have such a decomposition, in which case the following does not apply.

Lemma 4.12

ahah. okay. say you have u and v. Then the action of $ad(u)$ and $ad(v)$ splits over $\bigoplus_\alpha L_\alpha$:

(a) $x, y \in H \Rightarrow \langle x, y \rangle = \sum_{\alpha \in \Phi} m_\alpha \, \alpha(x) \alpha(y)$

(b) $[L_\alpha, L_\beta] \subseteq L_{\alpha + \beta}$

(c) $\langle \cdot, \cdot \rangle$ restricted to H is non-degenerate

(d) If α, β weights and if $\alpha + \beta \neq 0$, then $\langle L_\alpha, L_\beta \rangle = 0$.

(e) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

(f) α weight $\Rightarrow L_\alpha \cap L_{-\alpha}^{\perp} = 0$

(g) If $0 \neq h \in H$, then $\alpha(h) \neq 0$ for some $\alpha \in \Phi$. So Φ spans H^*, dual space of H.

proof: ⓐ Choose a basis for each weight space L_α, and take union to give a basis of L. Then $ad(x)$, $ad(y)$ are both represented by diagonal matrices.
Take $tr(ad(x)ad(y))$ to get ⓐ

$L = \bigoplus_{\alpha \in \Phi} L_\alpha$, and for each L_α we can choose a basis. Combining gives a basis of L. Now we know that if $u, v \in H$, then the action of $ad(u), ad(v)$ on L splits over this direct sum as:

$$\text{e.g.} \quad ad(u)(L) = \bigoplus_{\alpha \in \Phi} ad(u)(L_\alpha)$$

And the whole point is that on L_α, since $u \in H$, $ad(u)(x) = \alpha(x)$ $\forall x \in L_\alpha$. on a basis for L_α then, the map $ad(u)$ becomes just a diagonal matrix: e.g. if x, y is a basis for L_α, write $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and then

$$ad(u)(x) = \alpha(u)x = \begin{pmatrix} \alpha(u) \\ 0 \end{pmatrix} \text{ and } ad(u)(y) = \alpha(u)y = \begin{pmatrix} 0 \\ \alpha(u) \end{pmatrix} \text{ so } ad(u) = \begin{pmatrix} \alpha(u) & 0 \\ 0 & \alpha(u) \end{pmatrix}, \text{ and}$$

we can combine this idea to ive a diagonal matrix on all of L. Hence the result follows.

ⓑ Similar argument to (4.2)(i)

let $x \in L_\alpha$, $y \in L_\beta$, then

$$\begin{aligned} ad(h)[x,y] &= [ad(h)x, y] + [x, ad(h)y] \\ &= [\alpha(h)x, y] + [x, \beta(h)y] \\ &= \alpha(h)[x,y] + \beta(h)[x,y] \\ &= (\alpha+\beta)(h)[x,y] \Rightarrow [x,y] \in L_{\alpha+\beta} \end{aligned}$$

ⓒ (4.8)(c): The restriction of the killing form $\langle \cdot, \cdot \rangle_{ad}$ of L to H is also nondegenerate.

ⓓ Similar proof to that for (4.10)(a)

If $\alpha + \beta \neq 0$, then $\exists h \in H$ s.t $(\alpha+\beta)(h) \neq 0$. Then we just use basic properties of $\langle \cdot, \cdot \rangle$:

$$\alpha(h)\langle x, y \rangle = \langle \alpha(h)x, y \rangle = \langle ad(h)x, y \rangle = \langle [h,x], y \rangle = -\langle [x,h], y \rangle = -\langle x, [h,y] \rangle$$
$$= -\langle x, ad(h)y \rangle = -\langle x, \beta(h)y \rangle = -\beta(h)\langle x, y \rangle$$
$$\Rightarrow (\alpha(h) + \beta(h))\langle x, y \rangle = 0 \Rightarrow \langle x, y \rangle = 0.$$

the only case excluded in ⓓ was $\beta = -\alpha$.

ⓔ Suppose $\alpha \in \Phi$, but $-\alpha \notin \Phi$. Then $\langle L_\alpha, L_\beta \rangle = 0$ for all weights β by ⓓ
But $\langle \cdot, \cdot \rangle$ nondegenerate on L (3.5) and so $L_\alpha = 0$ ↯.

Suppose $\alpha \in \Phi$ and $-\alpha \notin \Phi$. By ⓓ, for all weights β (including α) we have $\langle L_\alpha, L_\beta \rangle = 0$, $\alpha + \beta \neq 0$. Now if $L_\alpha \notin \Phi$, then $L_\alpha = 0$ and $\Rightarrow \langle L_\alpha, L_\beta \rangle = 0$ $\forall \beta$ so that actually $\langle L_\alpha, L \rangle = 0$. But $\langle \cdot, \cdot \rangle$ is nondegenerate $\Rightarrow$ $L_\alpha = 0$ ↯, since α is a root.

ⓕ Take $x \in L_\alpha \cap L_{-\alpha}^{\perp}$. Then $\langle x, L_\beta \rangle = 0$ $\forall$ weights β. So $\langle x, L \rangle = 0$ and so $x = 0$ by nondegeneracy.

$x \in L_\alpha \cap L_{-\alpha}^{-1}$

$\langle x, L_\beta \rangle = 0$ $\forall \beta \neq -\alpha$ | $\langle x, L_\beta \rangle = 0$ $\forall \beta \neq \alpha$

$= \forall \beta.$

⑨ If $\alpha(h) = 0$ $\forall \alpha \in \Phi$, and $x \in H$. Then $\langle h, x \rangle = \sum_{\alpha \in \Phi} m_\alpha \, \alpha(h)\alpha(x) = 0$ $\forall x \in H$.

The nondegeneracy of $\langle \cdot, \cdot \rangle$ restricted to H then implies that $h = 0$. We've actually proved the contrapositive to ⑨, so :

If $h \neq 0$, $\Rightarrow \exists \alpha \in \Phi$ s.t $\alpha(h) \neq 0$

Example : $sl_3 = \{$trace zero, 3x3 matrices$\}$. Let H be a Cartan Subalgebra of trace zero diagonal matrices. Then $\dim H = 2$.

$$H = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

trace = 0

Can choose two entries on the diagonal, which uniquely determines the third one.

$H = L_0 = \{ x \in L : ad(h)(x) = 0 \ \forall \ h \in H \}$

$\Leftrightarrow L_{0,y} = \{ x \in L : (ad(y))^r(x) = 0 \}$.

H is a maximal abelian subalgebra, and so this restricts elements of H to have zeroes off the diagonal. We can then see that the only other condition on the matrix is that it has to be traceless, and for that we can choose a_{11}, a_{22}, and these determine the third diagonal entry $(-a_{11} - a_{22})$. So $\dim H = 2$.

Dfn 4.13 The α-string through β is the largest arithmetic progression

$$\beta - q\alpha, \ldots, \beta, \ldots, \beta + p\alpha$$

such that they are all weights.

linear forms $\beta + i\alpha$ such that $L_{\beta + i\alpha} \neq 0$.

α is a root,
β is a weight

remember α, β are linear forms. These are: $\beta + \alpha i$, where $i \in \{-q, \ldots, p\}$.

lemma 4.14: For $\alpha \in \Phi$, β weight, p, q as above. Then

ⓐ $$\beta(x) = -\left(\frac{\sum_{r=-q}^{p} r m_{\beta + r\alpha}}{\sum_{r=-q}^{p} m_{\beta + r\alpha}} \right) \alpha(x) \quad \text{for } x \in [L_\alpha, L_{-\alpha}].$$

ⓑ if $0 \neq x \in [L_\alpha, L_{-\alpha}]$, then $\alpha(x) \neq 0$

ⓒ $[L_\alpha, L_{-\alpha}] \neq 0$.

proof:

$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$, and above + below this the terms vanish

if $x \in L^{(1)} \Rightarrow tr(ad(x)) = 0$
Since $x = [y,z]$, and
$tr(ad(x)) = tr(ad([y,z]))$
$= tr([ad\,y, ad\,z])$
$= tr(ad\,y\,ad\,z - ad\,z\,ad\,y)$
$= tr(ad\,y\,ad\,z) - tr(ad\,z\,ad\,y)$
$= 0$.

ⓐ Let $M = \sum_{r=-q}^{p} L_{\beta + r\alpha}$ be a subspace of L. So $[L_\alpha, M] \subseteq M$, $[L_{-\alpha}, M] \subseteq M$.

by the above.

Let U = Lie subalgebra generated by L_α and $L_{-\alpha}$. Then $ad(U)(M) \subseteq M$.
Take $x \in [L_\alpha, L_{-\alpha}]$, then $x \in U^{(1)}$. So $ad(x)|_M : M \to M$ has trace 0.

But $tr\,ad(x)|_M = \sum_{r=-q}^{p} m_{\beta+r\alpha}(\beta + r\alpha)(x)$. Hence $\sum_{r=-q}^{p} m_{\beta+r\alpha}(\beta+r\alpha)(x) = 0$. Rearranging gives ⓐ
(see proof of lemma 4.12 a for argument + fact that $x \in [L_\alpha, L_{-\alpha}] \subseteq L_{\alpha-\alpha} = L_0 = H$

ⓑ If $x \neq 0 \in [L_\alpha, L_{-\alpha}]$ and $\alpha(x) = 0$, then from ⓐ, $\beta(x) = 0$ for any $\beta \in \Phi$. But 4.12ⓖ implies that $x = 0$ ↯.
ⓖ says if $0 \neq h \in H$, then $\exists\, \alpha \in \Phi$ s.t $\alpha(h) \neq 0$.

So $\alpha(x) \neq 0$ if $x \neq 0 \in [L_\alpha, L_{-\alpha}]$

otherwise $\alpha = 0$ map and result is trivial

ⓒ For $0 \neq v \in L_{-\alpha}$, we have $[h, v] = -\alpha(h)v$ for any $h \in H$ by definition. Choose $u \in L_\alpha$, $v \in L_{-\alpha}$ with $\langle u, v \rangle_{ad} \neq 0$ by 4.12 ⓕ. And choose $h \in H$ such that $\alpha(h) \neq 0$.
Set $x := [u, v] \in [L_\alpha, L_{-\alpha}]$, then $\langle x, h \rangle = \langle u, [v,h] \rangle$ since $\langle\, ,\, \rangle$ is invariant.
$= \alpha(h)\langle u, v\rangle$
$\neq 0$

So $x \neq 0$. ∎

4.12 f) says that for any $u \in L_\alpha$, $\exists\, v \in L_{-\alpha}$ s.t $\langle u, v \rangle \neq 0$
(otherwise $\langle u, v\rangle = 0$ $\forall v \in L_{-\alpha}$, $\Rightarrow u \in (L_{-\alpha})^\perp$
$\Rightarrow L_\alpha \cap (L_{-\alpha})^\perp \neq 0$ ↯)

just have to show that $\exists\, h \in L$ s.t $\langle x, h \rangle \neq 0$ to see that $x \neq 0$. Since if $x = 0$, $\langle x, h \rangle = 0$ $\forall h \in L$.

Lemma 4.15

set of roots

ⓐ $m_\alpha = 1$ $\forall \alpha \in \Phi$, and if $n\alpha \in \Phi$ for some $n \in \mathbb{Z}$, then $n = \pm 1$.

ⓑ $\beta(x) = \frac{q-p}{2}\alpha(x)$ $\forall x \in [L_\alpha, L_{-\alpha}]$.

ⓒ if $\dim(L) = n$, $\dim(H) = r$, then # of roots $= 2s = n - r$, and $r \leq s$.

proof: $u \in L_\alpha$, $v \in L_{-\alpha}$ s.t. $\langle u, v\rangle_{ad} \neq 0$ and $x = [u, v]$.
ⓐ Take u, v, x as in the proof of 4.14 ⓒ, and let B = Lie subalgebra generated by u, v.
N = subspace = span of v, H and $\sum_{r>0} L_{r\alpha}$.

$[u, N] \subseteq H + \sum_{r>0} L_{r\alpha} \subseteq N$ remember $x \in [u,v] \subseteq [L_\alpha, L_{-\alpha}] \subseteq L_{\alpha-\alpha} = L_0 = H$.

$[v, N] \subseteq [v, H] + \sum_{r>0}[v, L_{r\alpha}] \subseteq N$

So $[B, N] \subseteq N$. Then $x = [u, v] \in B^{(1)}$. Consider $ad(x)|_N : N \to N$. see previous note to see why $tr(ad(x)) = 0$.

$$0 = tr\,ad(x)|_N = -\underset{\substack{\downarrow \\ v}}{\alpha(x)} + \underset{\substack{\downarrow \\ H}}{0} + \sum_{r>0} \underset{\substack{\downarrow \\ \sum_{r>0} L_{r\alpha}}}{m_{r\alpha}\, r\alpha(x)}.$$
} using α-string thing restricted to N.

remember $m_\alpha = \dim \geq 0$

But $\alpha(x) \neq 0$ by 4.14 ⓑ, so $\sum_{r>0} r\, m_{r\alpha} = 1$ $\forall \alpha \in \Phi$. Thus $m_\alpha = 1$ and $r\alpha$ is a root for $r > 0$
$\Rightarrow r = +1$ (use the fact that its a positive sum and we know $m_\alpha \geq 1$ since $0 \neq u \in L_\alpha$)
Repeating for $-\alpha$, we get $r\alpha$ is a root for $r < 0$ $\Leftrightarrow r = -1$.

ⓑ Follows from ⓐ and 4.14

$$\beta(x) = -\left(\frac{\sum_{r=-q}^{p} r m_{\beta+r\alpha}}{\sum_{r=-q}^{p} m_{\beta+r\alpha}}\right)\alpha(x) \quad \text{for } x \in [L_\alpha, L_{-\alpha}].$$

$$\text{So } \beta(x) = -\left(\frac{\sum_{r=-q}^{p} r(1)}{\sum_{r=-q}^{p} 1}\right)\alpha(x) = -\left(\frac{p-q}{2}\right)\alpha(x) = \left(\frac{q-p}{2}\right)\alpha(x)$$

$\square$

ⓒ Follows from ⓐ and 4.12 ⓖ

We have a decomposition $L = L_0 \oplus L_\alpha$, and $\dim(L) = \underset{\substack{\shortparallel \\ H}}{\dim(L_0)} + \sum_{\alpha\in\Phi} \dim(L_\alpha)$

Now $L_0 = H \Rightarrow \dim(L_0) = \dim(H) = r$. Also $\dim(L_\alpha) = 1$ for every root, and if $\alpha \in \Phi$ then $-\alpha \in \Phi$
$\Rightarrow \sum \dim(L_\alpha) = 2s$, where $s = \#$ of "positive" roots. So

$$n = r + 2s \Rightarrow 2s = n - r$$

Remark: The Lie algebra B in the proof $\cong sl_2$ and you can use representation theory of sl_2 to prove the last two lemmas.

Lemma 4.16: if $\alpha \in \Phi$ and $c\alpha \in \Phi$ with $c \in \mathbb{C}$, then $c = \pm 1$.

proof: Set $\beta = c\alpha$. Take $\beta - q\alpha, \dots, \beta, \dots, \beta + p\alpha$ to be the α-string through β. Choose $0 \neq x \in [L_\alpha, L_{-\alpha}]$. Then $\alpha(x) \neq 0$ by 4.14 (b). Then $\beta(x) = (q-p)/_2\,\alpha(x)$ by 4.15 (b). So $c = \frac{q-p}{2}$.

If $q-p$ is even, then we're done by previous lemma. (which deals with $c \in \mathbb{Z}$)
If $q-p$ is odd, then take $r = \frac{1}{2}(p-q+1) \in \mathbb{Z}$ and $-q \leq r \leq p$. So $\beta + r\alpha$ is in the α-string through β. Thus, $\frac{1}{2}\alpha$ is a weight since $\beta + r\alpha = \frac{1}{2}\alpha$.

So Φ contains $\frac{1}{2}\alpha$, $2(\frac{1}{2}\alpha)$ ↯ by Lemma 4.15 ⓐ.

$\square$

Define for each $h \in H$, h^* by $h^*(x) = \langle h, x\rangle_{ad}$ $\forall x \in H$. Thus $h^* \in H^*$, and $h \longmapsto h^*$ is linear (by linearity of $\langle \cdot,\cdot \rangle_{ad}$). The map is injective by nondegeneracy. Hence, its surjective (finite dim. spaces), and we can write h_α for the preimage of $\alpha \in H^*$.

We can now define a symmetric bilinear form on H^*.

Dfn 4.17: $(\alpha, \beta) := \langle h_\alpha, h_\beta \rangle_{ad}$ for $\alpha, \beta \in H^*$.

Where $\begin{cases} \langle h_\alpha, x\rangle = \alpha(x) \\ \langle h_\beta, x \rangle = \beta(x) \end{cases} \forall x \in H$

$h^*(x) = \langle h, x \rangle_{ad}$

h_α is the unique element $\in H$ s.t $\alpha = h_\alpha^*$.
$= \langle h_\alpha, - \rangle_{ad}$

Lemma 4.18 : we can choose e_α, $e_{-\alpha}$ with $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$ so that $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\langle e_\alpha, e_{-\alpha} \rangle_{ad} = 1$.

proof: For $x \in H$, $\langle [e_\alpha, e_{-\alpha}], x \rangle = \langle e_\alpha, [e_{-\alpha}, x] \rangle$ by invariance

$= \alpha(x) \langle e_\alpha, e_{-\alpha} \rangle \longleftarrow$ $\left\{ \begin{array}{l} e_{-\alpha} \in L_{-\alpha} = \{ x \in L : ad(h)(x) = -\alpha(h)x \ \forall h \in H \} \\ \Rightarrow ad(x)(e_{-\alpha}) = -\alpha(x) e_{-\alpha} \\ \quad [x, e_{-\alpha}] = -\alpha(x) e_{-\alpha} \\ \Leftrightarrow \quad [e_{-\alpha}, x] = \alpha(x) e_{-\alpha}. \end{array} \right.$

$= \alpha(x) \longleftarrow$ can choose $e_\alpha, e_{-\alpha}$ s.t $\langle e_\alpha, e_{-\alpha} \rangle_{ad} = 1$ wlog

$= \langle h_\alpha, x \rangle \longleftarrow$ by uniqueness

Lemma 4.19 : For $\alpha, \beta \in \Phi$,

$\langle \cdot, \cdot \rangle$ = Killing form on L restricted to H
$\langle h_\alpha, x \rangle = \alpha(x) \ \forall x \in H$

(a) $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} = \dfrac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle} \in \mathbb{Z}$

(b) $4 \sum_{\beta \in \Phi} \dfrac{\langle h_\beta, h_\alpha \rangle^2}{\langle h_\alpha, h_\alpha \rangle^2} = \dfrac{4}{\langle h_\alpha, h_\alpha \rangle} \in \mathbb{Z}$

(c) $\langle h_\alpha, h_\beta \rangle \in \mathbb{Q} \quad \forall \alpha, \beta \in \Phi$.

(d) $\forall \alpha, \beta \in \Phi$, $\quad \beta - \dfrac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle} \alpha \in \Phi$.

and the corresponding statements wrt (α, β)

(b) if $0 \neq x \in [L_\alpha, L_{-\alpha}]$, then $\alpha(x) \neq 0$

proof: Consider $\langle h_\alpha, h_\alpha \rangle = \alpha(h_\alpha) \neq 0$ by 4.14 (b). Hence,

(a) $$\frac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle} = \frac{2\beta(h_\alpha)}{\alpha(h_\alpha)} = \frac{2(q-p)}{2} \in \mathbb{Z}$$ for α-string through β, $(-q, p)$.

(b) For $x, y \in H$, $\langle x, y \rangle = \sum_{\beta \in \Phi} \beta(x)\beta(y)$ by 4.12 a and 4.15a. So

$\langle h_\alpha, h_\alpha \rangle = \sum_{\beta \in \Phi} \beta(h_\alpha)^2 = \sum_{\beta \in \Phi} \langle h_\beta, h_\alpha \rangle^2$ by dfn of h_β.

So this says that $\langle h_\alpha, h_\alpha \rangle = \sum_\beta \langle h_\beta, h_\alpha \rangle^2 \Rightarrow \dfrac{\sum_\beta \langle h_\beta, h_\alpha \rangle^2}{\langle h_\alpha, h_\alpha \rangle} = 1$

Hence, we can multiply both sides to show:

$$4 \frac{\sum \langle h_\beta, h_\alpha \rangle^2}{\langle h_\alpha, h_\alpha \rangle^2} = \frac{4}{\langle h_\alpha, h_\alpha \rangle}$$

and in fact $4 \dfrac{\sum \langle h_\beta, h_\alpha \rangle^2}{\langle h_\alpha, h_\alpha \rangle^2} = \sum \Big(\underbrace{\dfrac{2\langle h_\beta, h_\alpha \rangle}{\langle h_\alpha, h_\alpha \rangle}}_{\in \mathbb{Z}} \Big)^2 \in \mathbb{Z}$.

(c) immediate from (a) and (b)

(d) $\beta - \frac{2\langle h_\beta, h_\alpha\rangle}{\langle h_\alpha, h_\alpha\rangle}\alpha = \beta + \frac{p-q}{2}\alpha$ from proof of (a)

I think its supposed to say $\beta + \frac{2(p-q)}{2}\alpha$ but still $\beta + (p-q)\alpha \in \alpha$-string through β.

Note that $\beta + \frac{p-q}{2} \in \alpha$-string through $\beta \in \Phi$. $\square$

Define $\tilde{H} = \mathbb{Q}$-span of $\{h_\alpha : \alpha \in \Phi\} \subseteq H$. Since $\{h_\alpha : \alpha \in \Phi\}$ span the complex vector space, we can take a subset $\{h_1, \dots, h_r\}$ that form a complex basis of H ($r = \dim H$). $r \leq s$)

Lemma 4.20: The killing form restricted to $\tilde{H}$ is an inner product, and $h_1, \dots, h_r$ is a $\mathbb{Q}$-basis of $\tilde{H}$.

proof: The form $\langle \cdot, \cdot \rangle$ is symmetric and bilinear, and has rational values on $\tilde{H}$ by 4.19 (c).
Let $x \in \tilde{H}$. Then

$$\langle x, x\rangle = \sum_{\alpha\in\Phi}(\alpha(x))^2 \quad \text{by } 4.13\ (a).$$

$$= \sum_{\alpha\in\Phi}\langle h_\alpha, x\rangle^2$$

Each $\langle h_\alpha, x\rangle \in \mathbb{Q}$, and so $\langle x,x\rangle \geq 0$. We get equality only if each $\langle h_\alpha, x\rangle = \alpha(x) = 0$ for every $\alpha \in \Phi$.
Thus, $x = 0$. an inner product.

It remains to show that each h_α is a rational linear combination of $h_1, \dots, h_r$. But if

$$h_\alpha = \sum_{i=1}^{r} \lambda_i h_i \qquad \lambda_i \in \mathbb{C}$$

this is true because $\{h_i\}_{i=1,\dots,r}$ span $\mathbb{C}$-span $\{h_\alpha : \alpha\in\Phi\}$ by dfn. So we just need to check $\lambda_i \in \mathbb{Q}$, not just $\lambda_i \in \mathbb{C}$

$$\Rightarrow \langle h_\alpha, h_j\rangle = \sum_{i=1}^{r} \lambda_i \langle h_i, h_j\rangle \in \mathbb{Q} \text{ by } (4.19)(c)$$

Consider that the matrix $(\langle h_i, h_j\rangle)$ is a rational and nonsingular matrix, since $\langle \cdot, \cdot\rangle$ is nondegenerate.

Multiplying by the inverse of this rational matrix shows that all the λ_i are rational. $\square$

Now we can make similar statements concerning the $\mathbb{Q}$-span of the roots Φ, using the symmetric bilinear form $(\cdot, \cdot)$ on H^*. Note

$$H^* \supseteq \mathbb{Q}\text{-span of } \Phi,$$

and $(\cdot, \cdot)$ defines an inner product on $\mathbb{Q}$-span of Φ, and a subset Φ' of Φ that is a $\mathbb{C}$-basis of H^* is actually a $\mathbb{Q}$-basis of $\mathbb{Q}$-span of Φ.

5 ROOT SYSTEMS

Dfn 5.1: a subset Φ of a real Euclidean vector space E is a **finite root system** if

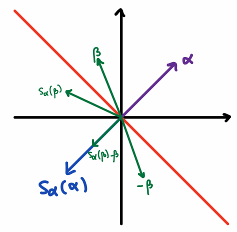
(i) Φ is finite, spanning E and not containing 0.

(ii) for each $\alpha \in \Phi$, there's a **reflection** S_α (preserving the inner product) with $S_\alpha(\alpha) = -\alpha$, the set of fixed points is a hyperplane of E, and S_α preserves Φ.

(iii) for each $\alpha, \beta \in \Phi$, $S_\alpha(\beta) - \beta$ is an integral multiple of α.

(iv) for $\alpha, \beta \in \Phi$, $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

(v) $S_\alpha(\beta) = \beta - \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \quad \forall \beta \in \Phi$



hyperplane = space orthogonal to α

Remark: 4.20 and the following discussion tells us that the roots of a finite dimensional, semisimple, complex Lie algebra give us a finite root system (with $E = \mathbb{R}$-span of the roots).

Dfn 5.2: The **rank** of a root system = $\dim E$.

Dfn 5.3: A root system is **reduced** if for each $\alpha \in \Phi$, the only roots proportional to α are $\pm\alpha$.

(4.16) $\Rightarrow$ root system from semisimple L is reduced.

Dfn 5.4: The **Weyl group** $W(\Phi)$ of a root system is a subgroup of the orthogonal group generated by the reflections S_α, $\alpha \in \Phi$. note: each element of Weyl group preserves Φ. But Φ generates H^* and so the weyl group can be seen as a subgroup of permutations of Φ, which is finite.

Dfn 5.5: for a finite root system, write $n(\beta,\alpha)$ for $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. Let $|\alpha| = (\alpha,\alpha)^{1/2}$. Then

$(\alpha,\beta) = |\alpha||\beta|\cos\phi$, where ϕ is an angle between α, β.

Then $n(\beta,\alpha) = \dfrac{2|\beta|}{|\alpha|}\cos\phi$.

Lemma 5.6: $n(\beta,\alpha)\,n(\alpha,\beta) = 4\cos^2\phi \in \mathbb{Z}$

proof: immediate. $\square$

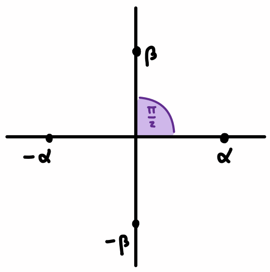
So $4\cos^2\phi$ can only take values $0, 1, 2, 3, 4$. Can only get 4 if α, β are proportional. Otherwise we have 7 possibilities.
iff (angle = 0)

Can have:

$n(\alpha,\beta)$	$n(\beta,\alpha)$	ϕ	Notes
0	0	$\frac{\pi}{2}$	
1	1	$\frac{\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
-1	-1	$\frac{2\pi}{3}$	$\lvert\beta\rvert = \lvert\alpha\rvert$
1	2	$\frac{\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
-1	-2	$\frac{3\pi}{4}$	$\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$
1	3	$\frac{\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$
-1	-3	$\frac{5\pi}{6}$	$\lvert\beta\rvert = \sqrt{3}\lvert\alpha\rvert$

$\Big\}$ possible reduced root systems

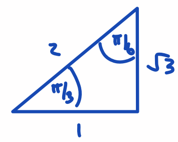
Example: reduced root systems of rank 2.

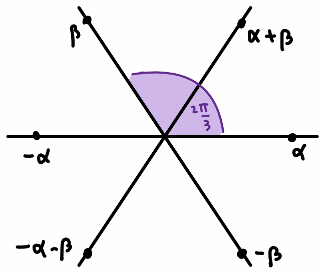


type $A_1 \times A_1$

arises from $sl_2 \times sl_2$.

Weyl group $= C_2 \times C_2$.



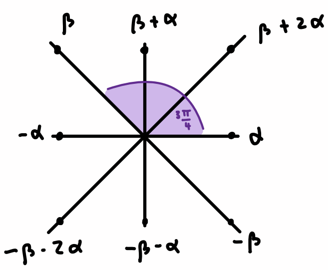


Type A_2

arises from sl_3.

Weyl group $= D_6$

$\dim(sl_3) = 8 = \underbrace{2}_{\text{rank}} + \underbrace{6}_{\text{\# of roots}}$



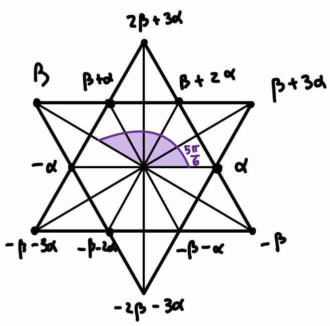
type B_2

α, β are different lengths

arising from sp_4 and so_5

Weyl group $= D_8$

$\dim = 2+8=10$



type G_2

arising from derivations of octonions

$\dim = 2+12 = 14$

Weyl group $= D_{12}$

α, β are different lengths.

These are the only reduced root systems of rank 2 up to isomorphism.

Dfn 5.7: An isomorphism of a root system

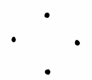
$$(E, \Phi) \longrightarrow (E', \Phi')$$ is a linear bijection

such that $\phi(\Phi) = \Phi'$

(note: ϕ need not be an isometry)

Dfn 5.8: ⓐ The direct sum of two root systems (E,Φ) and (E',Φ') is $(E\oplus E', \Phi \cup \Phi')$

ⓑ A root system that is not isomorphic to a direct sum of root systems is called **irreducible.**

E.g.  is reducible, since it is the direct sum of two root systems of rank 1.

Dfn 5.9: if $\alpha \in \Phi$, define the **co-root $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$**. Thus $(\alpha, \alpha^\vee) = 2$.

Exercise: if (E,Φ) is a root system, then (E,Φ') is a root system where $\alpha \in \Phi \Leftrightarrow \alpha^\vee \in \Phi^\vee$ (the dual of the root system).

Dfn 5.10: A root system is **simply laced** if all the roots are of the same length.

Example: the only irreducible, simply laced rank 2 root system is A_2.

Dfn 5.11: A subset Δ of a root system (E,Φ) is a **base of Φ** if

1) Δ is a vector space basis for E
2) each $\gamma \in \Phi$ can be written as a linear combination

$$\gamma = \sum_{\alpha\in\Delta} k_\alpha \alpha$$

with coefficients k_α integers and either **all ≥ 0 or all ≤ 0.**

Elements of Δ are called **simple roots**, and the γ where all $k_\alpha \geq 0$ are the **positive roots.** The set of such γ is denoted Φ^+. Similarly we define **negative roots** $(k_\alpha \leq 0)$ and Φ^-. Thus $\Phi = \Phi^+ \cup \Phi^-$.

(We'll see that a Δ always exist).

Example: in our 4 examples of rank 2, $\{\alpha,\beta\}$ form a base Δ.

Dfn 5.12: The **Cartan matrix** of a root system wrt. Δ is the matrix $(n(\alpha,\beta))_{\alpha,\beta\in\Delta}$

Example: Cartan matrix of G_2 $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

note: $n(\alpha,\alpha) = 2 \quad \forall \alpha \in \Phi$.

Dfn 5.13: A **coexeter graph** is a finite graph, each pair of vertices connected by 0, 1, 2 or 3 edges.
Given a root system Φ with base Δ, the coexeter graph of (E, Φ) wrt Δ has:

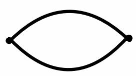
- vertices : elements of Δ = simple roots
- vertex α is joined to β for 0, 1, 2, 3 according to $n(\alpha,\beta)n(\beta,\alpha) = 0,1,2,3$

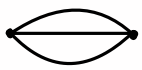
Example the Coxeter graphs of rank 1 and 2 reduced root systems:

rank 1 A_1 (sl_2)

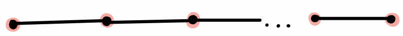
rank 2 $A_1 \times A_1$ $n(\alpha,\beta)\, n(\beta,\alpha) = 0\cdot 0 = 0$

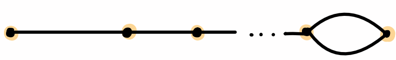
 A_2 $n(\alpha,\beta)\, n(\beta,\alpha) = (-1)\cdot(-1) = 1$

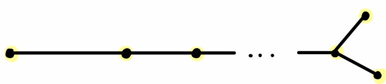
 B_2 $n(\alpha,\beta)n(\beta,\alpha) = (-1)\cdot(-2) = 2$

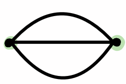
 G_2 $n(\alpha,\beta)n(\beta,\alpha) = (-1)\cdot(-3) = 3$

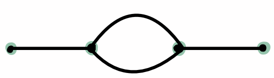
Theorem 5.14: Every connected, nonempty coexeter graph associated with a root system arising from a semisimple complex Lie algebra is isomorphic to

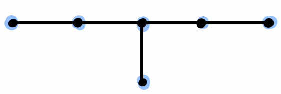
A_r  r vertices $r \geq 1$

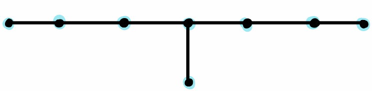
B_r  r vertices $r \geq 2$

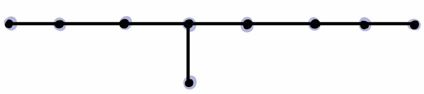
D_r  $r \geq 4$

G_2 

F_4 

E_6 

E_7 

E_8 

The Coexeter graphs are telling us the angles between the roots, but not their relative lengths.

B_r 

C_r 

arrow pointing towards shorter root.

e.g. B_2 from before 

G_2 

For G_2, F_4 its usual to include the arrow : F_4 

The graphs with arrows are called Dynkin Diagrams.

(we'll come back to the classification of Coexeter graphs when studying quivers, and we'll prove the Theorem for simply laced Coexeter graphs : A_r, D_r, E_6, E_7, E_8).

$$-\Phi^+(\gamma) = -\{\alpha\in\Phi : (\gamma,\alpha)>0\} = \{-\alpha\in\Phi : (\gamma,\alpha)>0\} = \{\alpha\in\Phi : (\gamma,-\alpha)>0\} = \{\alpha\in\Phi : (\gamma,\alpha)<0\}$$

For $\gamma\in E$, we can define $\Phi^+(\gamma) := \{\alpha\in\Phi : (\gamma,\alpha)>0\}$.

Consider $E\setminus\bigcup_{\alpha\in\Phi} P_\alpha$, where P_α is the hyperplane of s_α is non empty.

$P_\alpha = \{\gamma\in E : (\alpha,\gamma)=0\}$

Dfn 5.15 :

ⓐ γ is regular if $\gamma\in E\setminus\bigcup P_\alpha$, and thus $\Phi = \Phi^+(\gamma)\cup(-\Phi^+(\gamma))$. i.e. $(\gamma,\alpha)\neq 0\ \forall\alpha\in\Phi$

ⓑ $\alpha\in\Phi^+(\gamma)$ is indecomposable if it cannot be expressed as $\alpha=\alpha_1+\alpha_2$, $\alpha_1,\alpha_2\in\Phi^+(\gamma)$ ($\alpha_2\neq 0$).

Lemma 5.16: let $\gamma\in E$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable elements of $\Phi^+(\gamma)$ is a base of Φ. Every base has this form.

$\alpha\in\Phi^+(\gamma)\Rightarrow(\gamma,\alpha)>0\to$ bounded below + finite $\Rightarrow\exists$ min. guy.

proof a) each $\alpha\in\Phi^+(\gamma)$ is a non-negative, integral linear combination of elements of $\Delta(\gamma)$: otherwise, if we choose a 'bad' α, with (γ,α) minimal, so α decomposable, say $\alpha=\alpha_1+\alpha_2$, then $\alpha_i\in\Phi^+(\gamma)$. Then minimality $\Rightarrow$ α_1,α_2 good. ↯.

$(\gamma,\alpha) = (\gamma,\alpha_1+\alpha_2) = (\gamma,\alpha_1)+(\gamma,\alpha_2)>0$. But $\alpha_i\in\Phi^+(\gamma)\Rightarrow(\gamma,\alpha_1),(\gamma,\alpha_2)>0$. By assumption, α is the $\in\Phi^+(\gamma)$ with minimal (γ,α) that cannot be expressed as a non-neg, integral lin. comb. of $\Delta(\gamma)$. So α_1,α_2 must be good (non-neg, int. lin comb of $\Delta(\gamma)$). But then $\alpha=\alpha_1+\alpha_2$ is a non-neg int lin comb. of $\Delta(\gamma)$, a contradiction. ↯.

$\Delta(\gamma)$ spans $\Phi^+(\gamma)$ with $c_\alpha\geq 0$ and $-\Phi^+(\gamma)$ with $c_\alpha\leq 0$, so spans roots of E, and roots span E. So $\Delta(\gamma)$ spans E.

b) So $\Delta(\gamma)$ spans E and satisfies property (ii) for a base. We need to show linear independence. It's enough to show that when α,β distinct in $\Delta(\gamma)$, then $(\alpha,\beta)\leq 0$. We'll see how this follows in a little bit, but first let's show that $(\alpha,\beta)\leq 0$. Suppose $(\alpha,\beta)>0$. Then since $(\alpha,\beta)>0 \Leftrightarrow n(\alpha,\beta)>0$ looking at the table from above we must have that either $n(\alpha,\beta)=1$ or $n(\beta,\alpha)=1$ (or maybe both, but at least one). Wlog suppose $n(\beta,\alpha)=1$. Then notice that $s_\alpha(\beta)=\beta-n(\beta,\alpha)\alpha=\beta-\alpha$. But s_α permutes the roots Φ, and since β is a root $\Rightarrow s_\alpha(\beta)$ is a root $\Rightarrow\beta-\alpha$ is a root.

But we know that $\beta-\alpha\in\Phi^+(\gamma)$ or $-(\beta-\alpha)=\alpha-\beta\in\Phi^+(\gamma)$ since $\Phi=\Phi^+(\gamma)\cup(-\Phi^+(\gamma))$. If $\beta-\alpha\in\Phi^+(\gamma)$, then β is decomposable since $\beta=(\beta-\alpha)+\alpha$. Since $\beta\in\Delta(\gamma)$, ↯. Similarly $\alpha-\beta\in\Phi^+(\gamma)\Rightarrow\alpha$ decomposable. ↯.

Now to use this to show linear independence. Suppose $\sum r_\alpha\alpha=0$, $\alpha\in\Delta(\gamma)$ and $r_\alpha\in\mathbb{R}$. Separating the indices for which $r_\alpha>0$ and $r_\alpha<0$, we can write $\sum s_\alpha\alpha=\sum t_\beta\beta$, $\alpha\neq\beta$. Let $\varepsilon=\sum s_\alpha\alpha$. Then $(\varepsilon,\varepsilon)=\sum_{\alpha,\beta}s_\alpha t_\beta(\alpha,\beta)\leq 0$ since $(\alpha,\beta)\leq 0$. But $(x,x)\geq 0\ \forall x\in V$ by dfn of an inner product, so

$\Rightarrow(\varepsilon,\varepsilon)=0\Leftrightarrow\varepsilon=0$. Then $0=(\gamma,\varepsilon)=\sum s_\alpha(\gamma,\alpha)$, but $(\gamma,\alpha)>0$ since $\alpha\in\Delta(\gamma)\subset\Phi^+(\gamma)$,

$\Rightarrow s_\alpha=0\ \forall\alpha$. Similarly, all $t_\beta=0$. Hence $r_\alpha=0\ \forall\alpha$ and thus α's are linearly independent.

misses all the hyperplanes

possible since the intersection of "positive" open half-spaces associated with any basis of E is nonvoid.

Any base is of this form:

c) Now suppose Δ is a given base. Choose γ s.t $(\gamma,\alpha)>0$ $\forall\, \alpha \in \Delta$. So γ is regular. We'll show $\Delta = \Delta(\gamma)$. Certainly $\Phi^+ \subseteq \Phi^+(\gamma)$: we chose γ s.t $(\gamma,\alpha)>0$ $\forall\, \alpha\in\Delta$, and if $\beta \in \Phi^+$, then $\beta = \sum k_\alpha \alpha$ for $k_\alpha \geq 0$, so since $(\gamma,\alpha)>0$ $\forall\, \alpha \in \Delta$, and at least one $k_\alpha \neq 0$, then

$$\Rightarrow (\gamma,\beta) = \sum k_\alpha(\gamma,\alpha) > 0 \Rightarrow \beta \in \Phi^+(\gamma).$$

We also deduce similarly that $-\Phi^+ \subseteq -\Phi^+(\gamma) \Rightarrow \Phi^+(\gamma) \subseteq \Phi^+ \Rightarrow \Phi^+ = \Phi^+(\gamma)$.

But Δ is a base $\Rightarrow$ we can think of every element in Δ as a positive integral combination of Δ, and elements of Δ are indecomposable (basis for E) $\Rightarrow$ $\Delta \subseteq \Phi^+$ and in particular $\Delta \subseteq \Delta(\gamma)$.

But $|\Delta| = |\Delta(\gamma)| = \dim(E) \Rightarrow \Delta = \Delta(\gamma)$. $\square$

Lemma 5.17: For a base Δ of a reduced Φ,

(a) $(\alpha,\beta)\leq 0$, and so $\cos\phi \leq 0$, $n(\alpha,\beta)\leq 0$, and non diagonal entries in Cartan matrix are ≤ 0. for α,β distinct $\in \Delta$.

(b) if $\alpha \in \Phi^+$, and $\alpha \notin \Delta$, then $\exists\, \beta \in \Delta$ s.t $\alpha-\beta \in \Phi^+$

(c) Each $\alpha \in \Phi^+$ is of the form $\beta_1+\dots+\beta_n$ with each $\beta_1+\dots+\beta_i \in \Phi^+$ with each $\beta_i \in \Delta$.

(d) If α is simple $(\alpha\in\Delta)$, then s_α permutes $\Phi^+\setminus\{\alpha\}$ (+ve reflected to +ve except α ($\mapsto -\alpha$)).

Set $\rho = \frac{1}{2}\sum_{\beta\in\Phi} \beta$. Then $s_\alpha(\rho) = \rho - \alpha$.

remember $(\alpha,\beta)>0 \Rightarrow \alpha-\beta \in \Phi^+$ or $\beta-\alpha \in \Phi^+ \Rightarrow \alpha-\beta\in\Phi$.

Proof: any root must be a positive (≤ 0) or negative ($0 \geq$) linear combination of elements in the base.

(a) If $\alpha-\beta \in \Phi$, then this would contradict (ii) of the definition of the base. So $(\alpha,\beta)\leq 0$ follows (same argument as (b) in previous lemma).

(b) If $(\alpha,\beta)\leq 0$ $\forall\, \beta\in\Delta$, then $\Delta\cup\{\alpha\}$ would be linearly independent $\lightning$. So $(\alpha,\beta)>0$ for some β. Then $\alpha-\beta\in\Phi$ (same argument as before).

If $\alpha = \sum_{\gamma\in\Delta} k_\gamma \gamma$ with all $k_\gamma \geq 0$, then $k_\gamma>0$ for at least two $\gamma\in\Delta$ since $\alpha\notin\Delta$, so we know that $\alpha-\beta$ has at least one +ve coefficient. $\alpha-\beta\in\Phi$ $\Rightarrow$ forces $\alpha-\beta\in\Phi^+$.

(c) follows from (b) by induction. if $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, induct on $ht\,\beta := \sum_{\alpha\in\Delta} k_\alpha$

$\alpha-\beta\in\Phi$. now if

if $ht\,\beta = 1$, then $\beta=\alpha$ for some $\alpha\in\Delta$ and we're done. Suppose it holds for $k=n$. For $ht\,\beta = n+1$, by (b) $\exists\, \alpha\in\Delta$ such that $\beta-\alpha\in\Phi^+$. But by inductive hypothesis, write

$$\beta-\alpha = \beta_1+\dots+\beta_n \Rightarrow \beta = \beta_1+\dots+\beta_n+\alpha$$

$\square$

(d) If $\beta = \sum k_\gamma \gamma \in \Phi^+\setminus\{\alpha\}$, then $\exists\, k_\gamma>0$ with $\gamma\neq\alpha$. But coefficient of γ in $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is $k_\gamma>0$.

So all coefficients are ≥ 0 and so $s_\alpha(\beta)\in\Phi^+$. Hence, $\in \Phi^+\setminus\{\alpha\}$.

remember s_α by definition preserves Φ.

The last part with ρ follows.

$$s_\alpha(\rho) = \rho - \frac{2(\rho,\alpha)}{(\alpha,\alpha)}\alpha. \quad \text{But } 2(\rho,\alpha) = (\Sigma\beta,\alpha) = (\alpha,\alpha) + \Big(\sum_{\beta\in\Phi^+(\alpha)}\beta,\alpha\Big) - \Big(\sum_{\beta\in\Phi^+(\alpha)}\beta,\alpha\Big) = (\alpha,\alpha)$$

Lemma 5.18: Δ simple roots $\subseteq \Phi$.

a) if σ orthogonal $\in GL(E)$ and it satisfies $\sigma(\Phi)=\Phi$, then $\sigma s_\alpha \sigma^{-1} = s_{\sigma(\alpha)}$

b) Let $\alpha_1,\dots,\alpha_t \in \Delta$, not necessarily distinct. Write s_i for s_{α_i}.

If $s_t\cdots s_2(\alpha_1)$ negative (or $s_t\cdots s_1(\alpha_1)$ is positive), then for some $1\leq i\leq t$,

$s_t\cdots s_1 = s_t\cdots \hat{s}_i \cdots s_2\,\hat{s}_1$

c) If $\sigma = s_t\cdots s_1$ is an expression for an element of W with t minimal then $\sigma(\alpha_1)$ is negative.

pf: c) immediate from b)

formula for $S_{\sigma(\alpha)}$

a) $\alpha \in \Phi$, $\beta \in E$, then $\sigma s_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma s_\alpha(\beta) = \sigma(\beta - n(\alpha,\beta)\alpha) = \sigma(\beta) - n(\alpha,\beta)\sigma(\alpha)$.

So $\sigma s_\alpha \sigma^{-1}$ fixes the hyperplane $\sigma(P_\alpha)$ elementwise, and sends $\sigma(\alpha) \mapsto -\sigma(\alpha)$. Thus $\sigma s_\alpha \sigma^{-1} = s_{\sigma(\alpha)}$.

b) Take a minimal such expression with $s_t \cdots s_2(\alpha_1)$ negative. Then for $1 \le a < t$, $\beta_{a+1} = s_a \cdots s_2(\alpha_1)$ is positive by minimality. By 5.17 d) we have $\beta_t = \alpha_t$.

Let $\sigma = s_{t-1} \cdots s_2$. Then $s_t = s_{\sigma(\alpha_1)} = \sigma s_1 \sigma^{-1}$ by ⓐ. Result follows by rearranging. $\square$

Suppose that $s_t \dots s_2(\alpha_1)$ is negative, and this is an expression with minimal length. Then by minimality, $s_i \dots s_2(\alpha_1) \geq 0$ is positive $\forall\, i < t$: say $\beta_{i+1} := s_i \dots s_2(\alpha_1)$. Then β_{i+1} is positive for $i < t$, and negative when $i = t$. Note that $\beta_t = s_{t-1} \dots s_2(\alpha_1)$ is positive, but $\beta_{t+1} = s_t(\beta_t)$ is negative. Now s_t (generally reflections) permute $\Phi^+ \setminus \{\alpha_t\}$, so the only way that s_t maps something +ve to −ve is if $\beta_t = \alpha_t$. Let $\sigma = s_{t-1} \dots s_2$. Then (a) says $\sigma s_1 \sigma^{-1} = s_{\sigma(1)} = s_{\sigma(\alpha_1)} = s_{\alpha_t} = s_t$. Then rearrange.

orthogonal group generated by the reflections $\{ s_\alpha : \alpha \in \Phi \}$.

Lemma 5.19: $W = W(\Phi)$, Φ reduced.

a) If γ is regular $\in E$, then $\exists\, \sigma \in W$ with $(\sigma(\gamma), \alpha) > 0 \quad \forall\, \alpha \in \Delta$. I.e. W permutes the bases transitively. (go from one basis to another)

b) For $\alpha \in \Phi$, then $\sigma(\alpha) \in \Delta$ for some $\sigma \in W$.

c) $W = \langle s_\alpha \text{ for } \alpha \in \Delta \rangle$

d) If $\sigma(\Delta) = \Delta$ for $\sigma \in W$, then $\sigma = 1$.

proof: let $W' = \langle s_\alpha : \alpha \in \Delta \rangle \subset W$

First we'll prove a) and b) for W'.

W is finite so this is possible.

a) Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, and γ be regular. Choose $\sigma \in W'$ so that $(\sigma(\gamma), \rho)$ as large as possible. Then for $\alpha \in \Delta$, we have $s_\alpha \sigma \in W'$. So $(\sigma(\gamma), \rho) \geq (s_\alpha \sigma(\gamma), \rho)$ by maximality.

$$= (\sigma(\gamma), s_\alpha(\rho)) \quad \leftarrow s_\alpha \text{ preserves inner product}$$

$$\overset{5.17\ d)}{=} (\sigma(\gamma), \rho) - (\sigma(\gamma), \alpha)$$

$$\Leftrightarrow (\sigma(\gamma), \alpha) \geq 0$$

note that equality would imply $(\gamma, \sigma^{-1}\alpha) = 0 \Rightarrow \gamma \in P_{\sigma^{-1}(\alpha)}$, which contradicts regularity.

Also $\sigma^{-1}(\Delta)$ is a base with $(\gamma, \alpha') > 0 \quad \forall\, \alpha' \in \sigma^{-1}(\Delta)$. So the argument of 5.16 c) $\Rightarrow \sigma^{-1}(\Delta) = \Delta(\gamma)$. Since any base is of this form $\Delta(\gamma)$ by 5.16, transitivity on bases follows.

b) It suffices to show each root α is in a base, and then apply a). So, choose $\gamma_1 \in P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$.

Let $\varepsilon = \frac{1}{2} \min \{ |(\gamma_1, \beta)| : \beta \neq \pm\alpha \}$. Choose γ_2 with $0 < (\gamma_2, \alpha) < \varepsilon$, and $|(\gamma_2, \beta)| < \varepsilon$ for each $\beta \neq \pm\alpha$.

Define $\gamma = \gamma_1 + \gamma_2$. Then $0 < (\gamma, \alpha) < \varepsilon$, and $|(\gamma, \beta)| > \varepsilon$. So α is an indecomposable element of $\Phi^+(\gamma)$, and hence $\alpha \in \Delta(\gamma)$. $(\gamma_1, \alpha) = 0$ since $\gamma_1 \in P_\alpha$.

(c) In view of (b), it suffices to prove that each root belongs to at least one base. Since the only roots proportional to α are $\pm\alpha$, the hyperplanes P_β ($\beta \neq \pm\alpha$) are distinct from P_α, so there exists $\gamma \in P_\alpha$, $\gamma \notin P_\beta$ (all $\beta \neq \pm\alpha$) (why?). Choose γ' close enough to γ so that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for all $\beta \neq \pm\alpha$. Evidently α then belongs to the base $\Delta(\gamma')$.

c) It's enough to show for any root $\alpha \in \Phi$, $s_\alpha \in W'$. Find by b) some $\sigma \in W'$ with $\sigma(\alpha) \in \Delta$. Thus $s_{\sigma(\alpha)} \in W'$. But by 5.18 a), we saw $s_{\sigma(\alpha)} = \sigma^{-1} s_\alpha \sigma$, so $s_\alpha \in W'$.

d) Suppose d is false: $\exists\, \sigma \neq 1$ such that $\sigma(\Delta) = \Delta$. Write σ as a product of simple reflections in the shortest possible form. This contradicts 5.18 c). σ must be as short as it can be (i.e. $\sigma = id$).
$\sigma(\Delta) = \Delta$ sends +ve to +ve. If minimal, then 5.18 c says sends +ve to -ve.

Theorem 5.20: (not proved here) : $W(\Phi) = \langle \underbrace{s_\alpha}_{\text{simple reflections}} : s_\alpha^2 = 1,\ (s_\alpha s_\beta)^{\overbrace{m(\alpha,\beta)}^{\text{order of } s_\alpha s_\beta}} = 1 \rangle$

With $m(\alpha,\beta) = 2, 3, 4$ or 6 depending on the angle between α, β: $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$.

Construction of Root Systems from Cartan Matrix / Dynkin Diagrams.

Strategy: we'll need the following machinery

- $e_1, \dots, e_n$ orthonormal basis in Euclidean space.
- $I = \{$ integral combinations of $\frac{1}{2} e_i \}$
- J a subgroup of I
- x, y fixed reals > 0 with $\frac{x}{y} = 1, 2, 3$.

Define $\Phi = \{ \alpha \in J : \|\alpha\|^2 = x \text{ or } = y \}$

$E =$ span of Φ

We need that each s_α preserves length and $s_\alpha(\Phi) = \Phi$.

Note if $J \leq \sum \mathbb{Z} e_i$ and $\{x, y\} \subseteq \{1, 2\}$, then this is satisfied.

A_r, $r \geq 1$: Take $n = r+1$, and $J = (\sum \mathbb{Z} e_i) \cap \langle \sum_{i}^{n} e_i \rangle^\perp$. Let $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 2 \} = \{ e_i - e_j : i \neq j \}$
Then $\alpha_i = e_i - e_{i+1}$ are linearly independent, and if $i < j$, $e_i - e_j = \sum_{k=i}^{j-1} \alpha_k$. So $\{\alpha_i\}$ is a base for Φ

We know $(\alpha_i, \alpha_j) = 0$ unless $j = i, i+1$.
$(\alpha_i, \alpha_i) = 2$
$(\alpha_i, \alpha_{i+1}) = -1$

So Φ has Dynkin diagram of Type A_r. Each permutation of $1, \dots, r+1$ is an automorphism of Φ, and hence $W(\Phi) \cong S_{r+1}$:

s_{α_i} switches $i, i+1$, and we know $\{(i, i+1)\}$ generate S_{r+1}. This is the root system of $\mathfrak{sl}_{r+1}$.

B_r : $r \geq 2$: set $n = r$, $J = \{ \sum \mathbb{Z} e_i \}$, and $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 1 \text{ or } 2 \} = \{ \pm e_i, \pm e_i \pm e_j : i \neq j \}$

Let $\alpha_i = e_i - e_{i+1}$ for $i < r$, and $\alpha_r = e_r$. Then $e_i = \sum_{k=i}^{r} \alpha_k$, and $e_i + e_j$ is the sum of two such expressions, $e_i - e_j = \sum_{i}^{r-1} \alpha_k$, so basically $\alpha_1, \dots, \alpha_r$ is a base. This corresponds to a Dynkin diagram of type B_r associated with $\mathfrak{so}_{2r+1}$ (odd).

Action of Weyl group: $W(\Phi)$ gives all permutations and switching of signs of $\{e_1, \dots, e_r\}$. That is,
$W(\Phi) = \underbrace{C_2^r}_{\text{normal abelian subgroup}} \rtimes S_r$

C_r : $r \geq 3$: $n = r$, $J = \{ \sum \mathbb{Z} e_i \}$, $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 2 \text{ or } 4 \} = \{ \pm 2e_i, \pm e_i \pm e_j, i \neq j \}$, which is the dual of the system we had for B_r. Base is $\{ e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r, 2e_r \}$, and Weyl group is same as B_r, arising from Sp_{2r}.

D_r $r \geq 4$ $J = \{ \sum \mathbb{Z} e_i \}$, $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 2 \} = \{ \pm e_i \pm e_j \quad i \neq j \}$. Base = $\alpha_i = e_i - e_{i+1}$ $i < r$, and $\alpha_r = e_{r-1} + e_r$. Simple reflections cause permutations and an even number of sign changes.
$W(\Phi)$ = split extension of C_2^{r-1} by S_r permuting them (index 2 in group we had before)
Arises from SO_{2r}

E_8 : $n = 8$, set $f = \frac{1}{2} \sum_{i=1}^{8} e_i$, $J = \{ cf + \sum c_i e_i$: each $c, c_i \in \mathbb{Z}$ and $c + \sum c_i \in 2\mathbb{Z} \}$.
Then $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 2 \}$
$= \{ \pm e_i \pm e_j, \ i \neq j \} \cup \{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{k_i} e_i, \ \sum k_i \text{ even} \}$

Set $\alpha_1 = \frac{1}{2}(e_1 + e_8 - \sum_{2}^{7} e_i)$

$\alpha_2 = e_1 + e_2$, $\alpha_i = e_{i-1} - e_{i-2}$ for $i \geq 3$

E_7, E_6 : Take Φ from E_8 and intersecting with subspaces

$\Phi \cap \langle y \rangle^{\perp}$, $\Phi \cap \langle h, y \rangle^{\perp}$

for suitable h, y obtain $\alpha_1, \ldots, \alpha_7$ and $\alpha_1, \ldots, \alpha_6$ with Dynkin diagrams E_7, E_6.

F_4 : $n = 4$ set $h = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$, $J = \{ \sum \mathbb{Z} e_i + \mathbb{Z} h \}$,
$\Phi = \{ \alpha \in J : \|\alpha\|^2 = 1 \text{ or } 2 \} = \{ \pm e_i, \pm e_i \pm e_j \ i \neq j, \pm \frac{1}{2} e_1 \pm \frac{1}{2} e_2 \pm \frac{1}{2} e_3 \pm \frac{1}{2} e_4 \}$

Thus $e_2 - e_3$, $e_3 - e_4$, e_4, $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ form a basis.

G_2 : $n = 3$, $J = \sum \mathbb{Z} e_i \cap \langle e_1 + e_2 + e_3 \rangle^{\perp}$. Then $\Phi = \{ \alpha \in J : \|\alpha\|^2 = 2 \text{ or } 6 \}$
$= \{ \pm(e_i - e_j), 2e_i - e_j - e_k$ for i, j, k distinct$\}$

base : $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$.

6 REPRESENTATION THEORY OF SEMISIMPLE COMPLEX LIE ALGEBRAS

Theorem 6.1 (Weyl): Let L be a semisimple, finite dimensional Lie algebra, char$k = 0$. Then all finite dimensional representations are a direct sum of irreducible ones.

Definition 6.2: A representation is **completely reducible** if it is such a direct sum.

> Idea: say we have a repn $\rho : L \to \mathrm{End}(V)$, then ρ is said to be completely reducible if we can find some $W_1, \dots, W_n$ s.t $V = W_1 \oplus \cdots \oplus W_n$, and $\rho|_i : L \to \mathrm{End}(W_i)$ is a sub repn $\forall i$.

Lemma 6.3: The following are equivalent:

(1) all finite dimensional representations are completely reducible.

(2) whenever $\rho : L \to \mathrm{End}(V)$ with $W \subseteq V$ and $\dim(V/W) = 1$, and $\rho(L)(V) \subseteq W$ (in particular W is invariant), then there is a W' with $V = W \oplus W'$ and $\rho(L)(W') \subseteq W'$.

(3) The same as (ii) but with the restriction of ρ to W, $\rho_W : L \to \mathrm{End}(W)$, is irreducible.

pf: (1) ⇒ (2) ⇒ (3)

(3) ⇒ (2): Assume (3) to be true and prove (2) by induction on $\dim(V)$

If $W = 0$ or W is irreducible, we are done. So suppose $0 < U \leq W$ with $\rho(L)(U) \subseteq U$. Induction yields $W_1 \geq U$ with $V/U = W/U \oplus W_1/U$ and $\rho(L)(W_1) = W_1$. But $W_1/U \cong V/W$ as L-modules

Induction yields $W_1 = U \oplus W'$, hence $V = W \oplus W'$.

> If $W = 0$ then this is obvious. Same goes for $\dim W = 1$, since then W is clearly irreducible and so we can apply (3). Now suppose $\dim W > 1$. If W is irreducible, then by (3) we have (2). Now suppose that W is not irreducible. Then $\exists$ $0 < U < W$ s.t $\rho(L)(U) \subseteq U$. Now, $\dim\left(\frac{W}{U}\right) < \dim(W)$, and therefore we can apply our inductive hypothesis ⇒ $\frac{V}{U} = \frac{W}{U} \oplus \frac{W_1}{U}$ ⇒ $\frac{V}{W} \cong \frac{W_1}{U}$.
>
> Hence, $\rho(L)\left(\frac{W_1}{U}\right) \cong \rho(L)\left(\frac{V}{W}\right) = 0$ since $\rho(L)(V) \subseteq W$ by assumption. Hence ⇒ $\rho(L)(W_1) \subseteq U$. Now we know that $\dim U < \dim W$ by assumption, so $\dim\left(\frac{W_1}{U}\right) = \dim\left(\frac{V}{W}\right) = 1$. By induction we have that $W_1 = U \oplus W'$, and $W \cap W' = 0$. Also
>
> $$\begin{aligned} \dim(W) + \dim(W') &= \dim(W) + \dim(W_1) - \dim U \\ &= \dim W - \dim U + \dim W_1 - \dim U + \dim U \\ &= \dim V - \dim U + \dim U = \dim V \end{aligned}$$
>
> So actually $V = W \oplus W'$.

(2) $\Rightarrow$ (1): Suppose $\rho: L \to \mathrm{End}(A)$ with $B \subseteq A$, and $\rho(L)(A) \subseteq B$. Note change of letters, we're going to apply (2) to a different representation.

Let $\mu: L \to \mathrm{End}(\mathrm{End}(A))$; $x \mapsto (\theta \mapsto [\rho(x), \theta])$

Define $V = \{\theta \in \mathrm{End}(A) : \theta(A) \subseteq B, \ \theta|_B = \lambda 1_B$ for scalar $\lambda\}$, and $W = \{\theta \in \mathrm{End}(A) : \theta(A) \subseteq B, \ \theta|_B = 0\}$.

Note that $\dim V/W = 1$, and $\mu(L)(V) \subseteq W$.

↳ really just determined by scalar multiples.

$\forall x \in L$, $\mu(x) = (\theta \mapsto [\rho(x), \theta])$
so $\mu(x)(w') = [\rho(x), w'] = 0$ since $\mu(L)(w') \subseteq 0$.

Applying (2) gives complementary W' with $\mu(L)(W') \subseteq W'$, and so $V = W \oplus W'$.
$\exists$ $w' \in W'$ with $w'|_B = 1_B$. Also $\mu(L)(W')^{?} \subseteq W \cap W' = 0$. Thus w' commutes with $\rho(L)$ in $\mathrm{End}(A)$, and thus w' is a L-endomorphism of A $^{?}$; and $\rho(L)(\ker w') \subseteq \ker w'$. But $\ker w' \cap B = 0$ since $w'|_B = 1_B$. So if $a \in A$, then $w'(a) \in B$ and $w'(1 - w')a = 0$. So $(1-w')(a) \in \ker w'$ and
$a = w'(a) + (1-w')a \Rightarrow A = B \oplus \ker w'$. I.e. $\ker w'$ is complementary to B in A.

It suffices to show that if $\rho: L \to \mathrm{End}(V)$ is not irreducible, then $V = V_1 \oplus V_2$, $V_1, V_2 \neq 0$ as L-modules.
If ρ is not irreducible, $\exists$ $W < V$ such that $\rho(L)(W) \subseteq W$. Look at $\mathrm{End}(V)$ it gives an L-rep^n via $(\mathrm{ad} \circ \rho)$. Let $\varphi: L \to \mathrm{End}(\mathrm{End}(V))$, $L \mapsto \mathrm{ad}(\rho(L)) = [\rho(L), -]$.

$$\mathcal{U}_s = \{\theta \in \mathrm{End}(V) : \theta(V) \subseteq W, \ \theta|_W = \lambda 1_W \text{ for some } \lambda\}$$

$\mathcal{U}_s, \mathcal{U}_0 \subseteq \mathrm{End}(V)$.

$$\mathcal{U}_0 = \{\theta \in \mathrm{End}(V) : \theta(V) \subseteq W, \ \theta|_W = 0\}$$

look at $\mathrm{ad}(\rho(L))$ acting on $\mathrm{End}(V)$.

Then certainly $\mathcal{U}_0 \subseteq \mathcal{U}_s$ (0 is some λ). Since $\rho(W) \subseteq W$, $\mathcal{U}_s$ and $\mathcal{U}_0$ are fixed by $(\mathrm{ad} \circ \rho)$, in particular $(\mathrm{ad} \circ \rho)(L)(\mathcal{U}_s) \subseteq \mathcal{U}_0$

Clearly $\mathcal{U}_s/\mathcal{U}_0 = K$ (the field), so $\dim(\mathcal{U}_s/\mathcal{U}_0) = 1$. Hence we're in the position to apply 2. We get that in $\mathcal{U}_s$ there exists a complementary L-module $\mathcal{U}'$ to $\mathcal{U}_0$ such that $\mathcal{U}_s = \mathcal{U}_0 \oplus \mathcal{U}'$.

using $f \in \mathcal{U}_s$

Now certainly $\exists$ an endomorphism $f \in \mathcal{U}_s$ s.t wlog $f|_W = 1_W$, and since $f \notin \mathcal{U}_0$, it must be that $f \in \mathcal{U}'$. Now $\ker(f) \cap W = 0$ trivially. We also have that $f^2 = f$, since $\rho(L)(V) \subseteq W \Rightarrow f(V) \subseteq W$.
Hence $f(1_V - f) = 0 \Rightarrow (1_V - f)(v) \in \ker f$. Hence for any $v \in V$, we can write

$$v = \underbrace{f(v)}_{\in W} + \underbrace{v - f(v)}_{\in \ker f}$$

and so $\ker f$ is exactly a complementary L-module to W so that $V = W \oplus \ker f$ as L-modules.

$\square$

Proof of 6.1: We aim to show that 6.3 (iii) holds. Consider a representation $\rho: L \to \mathrm{End}(V)$, $W \subseteq V$, $\dim V/W = 1$, and $\rho(L)(V) \subseteq W$ with ρ_W irreducible. We aim to show W has a complementary invariant subspace.

Since quotients of semisimple Lie Algebras are semisimple (Corollary of 3.13), we may assume that $\ker\rho = 0$ (i.e. ρ is faithful). Let $\langle \cdot, \cdot \rangle_\rho$ be the trace form of ρ (see 3.2 a).

This is nondegenerate: let $L^\perp$ be the orthogonal space wrt $\langle \cdot,\cdot \rangle_\rho$. Then $L^\perp$ is an ideal by (3.3)(ii). Moreover $L^\perp$ is soluble since $tr(\rho(x)\rho(y)) = 0$ $\forall x,y \in L^\perp$, and we can apply Cartan's Solubility criterion to $\rho(L^\perp)$ to get $\rho(L^\perp)$ is soluble (but ρ is faithful).

But L is semisimple so all soluble ideals are zero. Using nondegeneracy, $\exists$ a basis $x_1, \dots, x_n$ of L $y_1, \dots, y_n$ of L such that $\langle x_i, y_j \rangle_\rho = \delta^i_j$. Define Casimir element of representation ρ by:

$$c = \sum_i \rho(x_i)\rho(y_i) \in End(V).$$

claim: c commutes with $\rho(L)$.

Thus $\ker c$ is invariant under $\rho(L)$. We'll show that $V = W \oplus \ker c$. Since $\rho(L)(V) \subseteq W$, we have $c(V) \subseteq W$. (from dfn of c). We're supposing that ρ_W is irreducible.

$$\rho(L)(c(W)) \subseteq c(W)$$

So $c(W) = W$ or 0 by irreducibility. But $c(W) = 0 \Rightarrow c^2 = 0$

$$\begin{aligned} \Rightarrow 0 &= tr(c) \\ &= tr(\textstyle\sum \rho(x_i)\rho(y_i)) \\ &= \textstyle\sum tr(\rho(x_i)\rho(y_i)) \\ &= \textstyle\sum \langle x_i, y_i \rangle_\rho \\ &= \dim V \\ &\neq 0 \quad ⚡ \text{ since } \operatorname{char} k = 0. \end{aligned}$$

So $c(W) = W$ and hence $\ker c \cap W = 0$. But $c(V) \subseteq W$ and so $\ker c \neq 0$. So $V = W \oplus \ker c$ as desired

Key points of proof:

(1) to show complete reducibility, want to show that if $\rho: L \to End(V)$ a repn with $W \subseteq V$ s.t $\rho(L)(V) \subseteq W$, $\dim(V/W) = 1$, and $\rho_W: L \to End(V)$ irreducible, then $\exists$ complementary W' s.t $V = W \oplus W'$ and $\rho(L)(W') \subseteq W'$.

(2) wlog assume ρ faithful. Show $\langle \cdot,\cdot \rangle_\rho$ nondegen since $\rho(L^\perp) \Rightarrow L^\perp$ soluble + L semisimple.

(3) $\langle \cdot,\cdot \rangle_\rho$ nondegen $\Rightarrow \exists$ bases $x_1, \dots, x_n$ & $y_1, \dots, y_n$ s.t $\langle x_i, y_j \rangle = \delta_{ij}$.

(4) Define Casimir elt: $c = \sum_i \rho(x_i)\rho(y_i) \in End V$.

(5) Show $\ker c$ is the W' we are looking for:
↳ Show $\rho(L)$ commutes with $c \Rightarrow \ker(c)$ preserved by $\rho(L) \Rightarrow \rho(L)(\ker c) \subseteq \ker(c)$.
↳ Show $\ker c \cap W = 0$. Then $c(V) \subseteq W$ + this fact $\Rightarrow V = \ker c \oplus W$.

Casimir Elements

$\rho: L \to \mathrm{End}(V)$, suppose $\langle \cdot, \cdot \rangle_\rho$ nondegenerate, e.g. as in proof of Weyl's Thm (6.1), or Killing form of semisimple Lie Algebras.

$\{x_1, \dots, x_n\}$ basis for L, and $\{y_1, \dots, y_n\}$ dual basis (for L) wrt nondegenerate form $\langle x_i, y_j \rangle_\rho = \delta_{ij}$.

Let $c = \sum_{i=1}^{n} \rho(x_i)\rho(y_i) \in \mathrm{End}(V)$

Lemma 6.4 $[c, \rho(z)] = 0 \quad \forall z \in L$. (commute in $\mathrm{End}(V)$).

$D(rs) = D(r)s + rD(s)$ (ring product)

pf: $[c, \rho(z)] = [\sum \rho(x_i)\rho(y_i), \rho(z)]$
$= \sum \rho(x_i)[\rho(y_i), \rho(z)] + \sum [\rho(x_i), \rho(z)]\rho(y_i)$

using fact that its a derivation
$[\rho(x)\rho(y), \rho(z)]$
$= [\cdot, \rho(z)](\rho(x)\rho(y))$
$= [\rho(x), \rho(z)]\rho(y) + [\rho(y), \rho(z)]\rho(x$

Write $[y_i, z] = \sum_{j=1}^{n} a_{ij} y_j$, and $[x_i, z] = \sum_{j=1}^{n} b_{ij} x_j$, then

$$a_{ij} = \langle [y_i, z], x_j \rangle_\rho = \langle [x_j, y_i], z \rangle_\rho \quad \text{by invariance of form}$$

and similarly $b_{ij} = \langle [x_i, z], y_j \rangle_\rho = \langle [y_j, x_i], z \rangle_\rho$ by invariance of form

$= -a_{ji}$

Hence $[c, \rho(z)] = \sum_{i,j} \rho(x_i)\rho(y_j)a_{ij} + \sum_{i,j} \rho(x_j)\rho(y_i)b_{ij} = 0$ $\square$

In fact, the definition of c is independent of choice of basis, but does depend on the trace form.

> Tools used: write $[c, \rho(z)]$, expand out. Then use fact $[\cdot, \rho(z)]$ is a derivation on ass. alg of $\mathrm{End}(V)$. Write $[x_i, z]$ and $[y_i, z]$ in terms of x_j and y_j respectively. Note that coefficients are antisymmetric, so the sum vanishes.

Universal Enveloping Algebra

The study of the representation theory of Lie Algebras is sometimes more easily understood by defining an associative algebra $U(L)$, known as the enveloping algebra of L.

Dfn 6.5: $U(L)$ is the associative algebra with generators $X\ (\in L)$ and relations $\underbrace{XY - YX}_{\text{commutator of } X,Y \text{ in enveloping algebra}} = \underbrace{[X,Y]}_{\text{lie bracket in } L}$. for $X,Y \in L$

This is equivalent to taking a basis $X_1, \dots, X_n$ of L and using generators X_i and relations $X_i X_j - X_j X_i = [X_i, X_j]$, together with linearity condition

$$\underbrace{\lambda X_i + \mu X_j}_{\in U(L)} = \underbrace{\lambda X_i + \mu X_j}_{\in L}$$

and addition in $U(L)$ is the same as in L.

Example: if L is abelian then $U(L) \cong \mathbb{C}[X_1, \dots, X_n]$ where $X_1, \dots, X_n$ is a basis of L (a polynomial algebra)

Because L abelian $\Leftrightarrow$ $[X,Y] = 0\ \forall X,Y$ $\Rightarrow$ $XY - YX = 0$ $\Rightarrow$ $XY = YX$ $\Rightarrow$ polynomial algebra

In general, you should view the enveloping algebra as a potentially noncommutative polynomial algebra.

Theorem 6.6 (Poincaré - Birkhoff - Witt PBW): $U(L)$ has a basis as a $\mathbb{C}$-vector space $\{X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} : m_i \in \mathbb{Z}_{\geq 0}\}$ where $X_1, \dots, X_n$ is a basis.

PBW is for a totally ordered basis $X_1 \leq X_2 \leq \dots \leq X_n$. Look at canonical monomials.

The reason for introducing the enveloping algebra is that there is a 1-1 correspondence between

$$\left\{ \begin{array}{l} \rho: L \to \mathrm{End}(V) \\ \text{representations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \bar{\rho} : U(L) \to \mathrm{End}(V) \\ V \text{ is a } U(L)\text{-module} \end{array} \right\}$$

$\bar{\rho} : U(L) \to \mathrm{End}(V)$. Then V is an abelian group with a representation of $U(L)$ over it. For $x \in U(L)$, the action of x is the map $x : v \mapsto \bar{\rho}(x)(v)$, which is enough to describe the module structure of V over $U(L)$.

Note: The Casimir elements we produced are images under $\bar{\rho}$ of an element of $U(L)$ of the form $\Sigma X_i Y_i$, where X_i are a basis and Y_i is a dual basis of L wrt the nondegenerate form

The proof of our lemma shows that these elements $\Sigma X_i Y_i$ are central in $U(L)$. ↳ commutes with $\bar{\rho}(z)\ \forall z \in U(L)$ In particular, if L is semisimple, and so the killing form is nondegenerate, we produce a central element $\Omega = \Sigma X_i Y_i$, where the bases are dual wrt. the killing form. Then $U(L)$ has a nontrivial centre assuming ρ is faithful $\Leftrightarrow$ $\bar{\rho}$ injective

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Now return to the representation theory of semisimple L. Take a CSA H, and roots Φ, and a base of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$. We have positive roots Φ^+. By the Cartan decomposition of L, we can consider the sum of the weight spaces corresponding to positive roots:

$$N = \sum_{\alpha \in \Phi^+} L_\alpha \quad , \text{ and } \quad N^- = \sum_{\alpha \in \Phi^+} L_{-\alpha}$$

Denote $B = H \oplus N$ the Borel subalgebra. Note that N is a nilpotent subalgebra, and so we have a decomposition

$$L = N^- \oplus H \oplus N = N^- \oplus B$$

For each $\alpha \in \Phi^+$, pick $x_\alpha \in L_\alpha$, and $y_\alpha \in L_{-\alpha}$, then $[x_\alpha, y_\alpha] \in L_{\alpha-\alpha} = L_0 = H$

Write L_i for L_{α_i} for simple roots α_i, x_i for x_{α_i}, etc...

Consider a representation $\rho: L \to \mathrm{End}(V)$

Dfn 6.7: Let $V_\omega = \{ v \in V : \rho(h)(v) = \omega(h) v \quad \forall h \in H \}$ be the weight space of weight ω, where $\omega \in H^*$.

This extends the definition of weight spaces of L arising from the adjoint representation to general representations. We define multiplicity to be the dimension of the weight space. The set of weights where $V_\omega \neq 0$ are the roots of V.

Notice that if $\dim(V)$ is finite, then $\exists$ weight spaces because H is abelian, and so there are common eigenspaces for H in V: the weight spaces.

Lemma 6.8

a) $\rho(L_\alpha) V_\omega \subseteq V_{\omega+\alpha}$ if $\omega \in H^*$, $\alpha \in \Phi$

b) The sum of the V_ω is direct and is invariant under $\rho(L)$.

c) (assuming L is semisimple) if $\dim(V) < \infty$ then $V =$ direct sum of weight spaces.

pf: a) For $x \in L_\alpha$, $v \in V_\omega$, $h \in H$, then
$$\begin{aligned}\rho(h)(\rho(x)(v)) &= \rho(x)\rho(h)(v) + \rho([h,x])(v)\\ &= \rho(x)(\omega(h)v) + \rho(\mathrm{ad}(h)(x))(v)\\ &= \omega(h)\rho(x)(v) + \rho(\alpha(h)x)(v) = (\omega(h)+\alpha(h))\rho(x)(v)\end{aligned}$$

b) The sum of common eigenspaces is always direct for commuting endomorphisms. The invariance comes from a).

Observe that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ - clearly invariant under $\rho(H)$.

> Don't know about the commuting endomorphisms thing, but you can show that if ω_1 and ω_2 are weights with $v \in V_{\omega_1} \cap V_{\omega_2}$, then $v = 0$.

c) If V were irreducible, then the direct sum of the weight spaces is a nonzero invariant subspace, and hence the whole of V. For general fin. dim. V, we can use Weyl's Thm so that V is a direct sum of irreducibles. $\square$

Dfn 6.9: v is a primitive element of weight ω if it satisfies

(i) $v \neq 0$, has weight ω ($\omega \in H^*$) $v \in V_\omega$

(ii) $\rho(x_\alpha)(v) = 0 \quad \forall \alpha \in \Phi$ He's an idiot. Wiki says that basically v is annihilated by N: $\rho(N)(v) = 0$

Condition (ii) is equivalent to $\rho(x_\alpha)(v) = 0 \quad \forall \alpha \in \Delta$.

If v is primitive, then $\rho(B)(v)$ is 1 dimensional. That is because $B = H \oplus N$, and $\rho(B) = \rho(H \oplus N) \overset{\rho(N)=0}{=} \rho(H)$ which is just scalar multiplication. Thus the primitive elements are the common eigenvectors for B.

$\Leftarrow$: if v is a common eigenvector for B, then its killed by $B^{(1)} = N$ (since H is a maximal abelian subalgebra) and so (ii) is satisfied.

> if v is a common eigenvector for B, then $\forall x \in B$, $\rho(x)(v) = \lambda_x v$ (where λ_x depends on our map). Then for any $x, y \in B$, $\rho([x,y])(v) = [\rho(x),\rho(y)](v) = \lambda_x\lambda_y v - \lambda_y\lambda_x v = 0$. So v is killed by $B^{(1)} = [H \oplus N, H \oplus N] = [N,N] \overset{!}{=} N$

Remark: Any finite dimensional V contains a primitive element by Lie's Theorem (2.18)

Proposition 6.10 Let v be a primitive element of weight ω, and let $W = \rho(L)(v)$. Then $y_i \in L_{-\beta_i}$

(i) W is spanned by $\rho(y_1)^{m_1} \cdots \rho(y_k)^{m_k}(v)$, where the β_i are the distinct +ve roots and $m_i \in \mathbb{Z}_{\geq 0}$

(ii) the weights of W are of the form $\omega - \sum_{i=1}^{r} p_i \alpha_i$, where $\{\alpha_1, \dots, \alpha_r\}$ is a base with $p_i \in \mathbb{Z}_{\geq 0}$, and they have finite multiplicity (weight spaces are finite dimensional)

(iii) ω has multiplicity 1, and the weight space in W of weight $\omega = \langle v \rangle$.

(iv) $\rho_W : L \to \mathrm{End}(W)$ is indecomposable. I.e. W cannot be expressed nontrivially in the form of a direct sum $W_1 \oplus W_2$, with W_i invariant.

Switch to talking about $\mathcal{U}(L)$-modules
$$\begin{cases} X_\alpha \longleftrightarrow x_\alpha \in L \\ Y_\alpha \longleftrightarrow y_\alpha \in L \\ H_\alpha \longleftrightarrow h_\alpha \end{cases}$$

proof: maybe wlog say $L \leq \mathrm{End}(V)$, with trivial repn.

(i) basis for $L := \{x_\alpha, y_\alpha, \text{basis of } H\}$. Then the PBW Theorem (6.6) says that

(4.15 (a) says L_α one dim)

$$\mathcal{U}(L) = \sum Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k} \, \mathcal{U}(B)$$

(and the sum is direct). Consider $W = \mathcal{U}(L)(v)$. But v is a common eigenvector for B, and so

$$W = \sum Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k} \langle v \rangle \qquad (\dagger)$$

(ii) By 6.8 a), $Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k}(v)$ has weight $\omega - \sum_{i=1}^{k} m_i \beta_i$. But each β_i is a positive integral combination of the simple roots. So this weight is of the form $\omega - \sum p_i \alpha_i$ with $p_i \in \mathbb{Z}_{\geq 0}$. Notice that $\omega - \sum p_i \alpha_i$ can only arise from finitely many $\omega - \sum m_i \beta_i$, and so the multiplicity of $\omega - \sum p_i \alpha_i$ is finite.

(iii) $\omega - \sum m_j \beta_j$ can only be ω if all the m_i are zero. So the only subspace in $\dagger$ of weight ω is $\langle v \rangle$. So ω weight space is $\langle v \rangle$, with multiplicity 1.

(iv) If $W = W_1 \oplus W_2$ with W_i nonzero, then $W_\omega = (W_1)_\omega \oplus (W_2)_\omega$. But W_ω is one dimensional, and so one of the $(W_i)_\omega = 0$, and v has to lie in the other. But v generates W, and so one summand will be the whole of W.

$\square$

Theorem 6.11: Let V (not necessarily finite dimensional) be a simple $\mathcal{U}(L)$-module (ρ is an irreducible representation) and suppose V contains a primitive element v of weight ω.

a) v is the only primitive element of V up to scalar multiplication.

b) The weights of V have the form $\omega - \sum p_i \alpha_i$ with $p_i \in \mathbb{Z}_{\geq 0}$. They have finite multiplicities, and ω has multiplicity 1, and V is a sum of the weight spaces.

c) For two simple modules V_1 and V_2, with primitive elements v_1 and v_2 of weight ω_1 and ω_2 respectively, then $V_1 \cong V_2$ iff $\omega_1 = \omega_2$.

Dfn 6.12: The weight ω of the primitive element V is known as the highest weight.

proof of 6.11: Apply 6.10. Since V is simple, $V = W = \mathcal{U}(L)v$, and so part b) follows.

a) Let v' be another primitive element of weight w'. Then $w' = w - \sum p_i \alpha_i$ for some $p_i \in \mathbb{Z}_{\geq 0}$. But also $w = w' - \sum q_i \alpha_i$ for some $q_i \in \mathbb{Z}_{\geq 0}$. This is only possible if $q_i = p_i = 0$
$\Leftrightarrow$ $w = w'$ and so v' must be a scalar multiple of V (the weight space of $w = \langle v \rangle$).

c) If $V_1 \cong V_2$, then $w_1 = w_2$ (the highest weight for both). Conversely, suppose $w = w_1 = w_2$. Set $V = V_1 \oplus V_2$ and $v = v_1 \oplus v_2$. The projection $\pi : V \to V_2$ induces a homomorphism $\pi|_W : W \to V_2$ under the action of $\mathcal{U}(L)$
$W = \mathcal{U}(L)v$. Note that $\pi(v) = v_2$, and v_2 generates V_2, so $\pi|_W$ is surjective.
Note $\ker \pi|_W = V_1 \cap W \subseteq V_1$. However, the only elements of weight w in V_1 are the scalar multiples of v_1. But $v \notin \ker \pi|_W$. So $\ker \pi_1 = V_1 \cap W$ does not contain any nonzero elements of weight w, and so $V_1 \cap W \subsetneq V_1$. By simplicity $V_1 \cap W = 0$, and so $\pi|_W$ is injective.
I.e. $W \cong V_2$ via $\pi|_W$. Similarly $W \cong V_1 \Rightarrow V_1 \cong V_2$.

Theorem 6.13: For each $w \in H^*$ there is a simple $\mathcal{U}(L)$-module of highest weight w.

Sketch of proof:

Return to $\mathfrak{sl}_2$ $\quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Proposition 6.15: Let V be a $\mathcal{U}(\mathfrak{sl}_2)$-module with primitive element v of weight w.
Set $e_n = \frac{1}{n!} Y^n(v)$, $e_1 = 0$.

Then (i) $He_n = w - 2ne_n$
(ii) $Ye_n = (n+1)e_{n+1}$
(iii) $Xe_n = (w - n + 1)e_{n-1}$ $\quad \forall n \geq 0$

pf: exercise

Corollary 6.16:

Either a) $(e_n)_{n \geq 0}$ are all linearly independent
b) the weight w of V is an integer $m \geq 0$, and the elements $e_1, \dots, e_m$ are linearly independent, and $e_{m+1} = 0$

pf: almost immediate

Corollary 6.17: if V is finite dimensional then we must be in case b) of 6.16. The subspace $e_1, \dots, e_n$ is invariant under L and we have the simple $\mathcal{U}(sl_2)$-modules we met earlier. The weights are $m, m-2, m-4, \dots, -m$, each with multiplicity 1.

Thm 6.18: let $\omega \in H^*$ and let V be the simple $\mathcal{U}(L)$-module of highest weight ω. Then

$$V \text{ is finite dimensional } \iff \forall \alpha \in \Phi^+, \quad \omega(h_\alpha) \in \mathbb{Z}_{\geq 0}$$

proof:
$\Rightarrow$ if v is a primitive element for L, then it is primitive for any of the subalgebras.

Remember $\langle X_\alpha, Y_\alpha, H_\alpha \rangle \cong sl_2 \quad \forall \alpha \in \Phi$.
But our knowledge of representations of sl_2
$\Rightarrow$ if V fin dim, then $\omega(h_\alpha) \in \mathbb{Z}_{\geq 0}$ (by 6.16/6.17).

$\Leftarrow$ not proved here.

Dfn 6.19: The weights satisfying the condition in Theorem 6.18 are <u>integral</u>. They are all positive integral combinations of <u>the fundamental weights</u> $:= \omega_i(h_j) = \delta_{ij}$.

(recall $h_{\alpha_i} =: h_i$ for simple roots α_i).

The irreducible representations with highest weight being a fundamental weight being a fundamental weight is a fundamental representation.

Example: sl_n roots are linear forms

$$\alpha_{ij} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \longmapsto \lambda_i - \lambda_j$$

$$\Sigma \lambda_i = 0 \qquad\qquad (i \neq j)$$

Base $\alpha_i = \alpha_{i,i+1}$

$$h_i = \begin{array}{c} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ i^{th} \;\; (i+1)^{th} \end{array} \qquad Y_i = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \qquad X_i = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

Fundamental weights

$$\omega_i(h) = \lambda_1 + \dots + \lambda_i$$

Where h is a diagonal matrix.

7 FINITE DIMENSIONAL ASSOCIATIVE ALGEBRAS

Example: $R = M_n(D)$ = $n \times n$ matrices over a division algebra D (e.g. $D = \mathbb{H}$)

right ideals are generated by a matrix A

Then $AR = \{ B : \text{Columns of } B \subseteq \text{right span of columns of } A \}$.

In general, a right ideal is of the form

$\{ B : \text{columns of } B \subseteq \text{right } D\text{-subspace of } D^n \}$

space of column vectors with D entries.

Similarly for the left ideals:

A left ideal is of the form: $\{ B : \text{rows of } B \subseteq \text{left } D\text{-subspace of span of row vectors} \}$

The only two sided ideals are 0 and $M_n(D)$. Thus $M_n(D)$ is a simple algebra.

Dfn 7.1: R is a simple (associative) algebra if its only two-sided ideals are 0 and R.

Example: kG, k field, G finite group:

k-vector space with basis labelled by group elements $g \in G$.

$$\left(\sum \lambda_g g\right)\left(\sum \mu_g g\right) = \sum \nu_g g \quad \text{where} \quad \nu_g = \sum_{hk=g} \lambda_h \mu_k$$

Dfn 7.2: The Jacobson radical $J(R)$ is the intersection of the maximal (proper) right ideals

Note: I is a maximal right ideal $\Leftrightarrow$ R/I is a simple right R-module.

$\Rightarrow$ Suppose that R/I is not a simple right R-module. Then $\exists$ a proper, nonzero submodule J. The preimage of J under the quotient map is a proper right ideal of R containing I. So I is not maximal.
$\Leftarrow$ Suppose I is not maximal. Then $\exists$ right ideal J of R containing I. The quotient map $q: R/I \to R/J$ then has a nontrivial kernel, which is also $\neq R/I$. But $\ker(q)$ is always a 2-sided ideal of R/I. So R/I is not simple. $\square$

Let M be a right R-module, and $m \in M$. Then the annihilator of m,

$$\mathrm{Ann}_R(m) : \{ r \in R : mr = 0 \}$$

is a right ideal of R, but not necessarily a 2-sided ideal. if $s \in R$, $r \in \mathrm{Ann}_R(m)$, then $mrs = 0s = 0 \Rightarrow rs \in \mathrm{Ann}_R(M)$.

However, $\mathrm{Ann}_R(M) = \bigcap_{m \in M} \mathrm{Ann}_R(m)$ (ann. of module) is a 2-sided ideal.

let $\mathcal{A} = \mathrm{Ann}_R(M)$. Let $r \in \mathcal{A}$, $s \in R$. Then $\forall m \in M$, $mr = 0 \Rightarrow (mr)s = 0 \Rightarrow m(rs) = 0 \ \forall m \in M \Rightarrow rs \in \mathcal{A} \Rightarrow$ right ideal
But M is a right module, so $\forall s \in R$, if $m \in M$ then $ms \in M$. So if $r \in \mathcal{A}$, then $\forall m \in M$, $(ms)r = 0 \Rightarrow m(sr) = 0$
$\Rightarrow sr \in \mathcal{A} \Rightarrow$ left ideal. $\square$

If M is simple, then $\mathrm{Ann}_R(m)$, $m \neq 0$ are maximal right ideals since $mR = M$. So we can see that

$$J(R) = \bigcap_{M \text{ simple right modules}} \mathrm{Ann}_R(M) = \text{2 sided ideal.}$$

$\square$

We can set up a map $f: R \to M$; $r \mapsto mr$. Then $mR = M$ since $m \neq 0$ and is simple, and mR is an ideal of M. In particular, $\ker(f) = \mathrm{ann}_R(m)$. Now $M \cong mR \cong R/\mathrm{Ann}_R(m)$. But M is simple, so $R/\mathrm{Ann}_R(m)$ is simple, so $\mathrm{Ann}_R(m)$ is a maximal right ideal

So it is clear then that $J(R) \subseteq \bigcap_{M \text{ simple}} \mathrm{Ann}_R(M)$. To see the reverse inclusion, let $a \in \bigcap \mathrm{Ann}_R(M)$, and let I be any maximal right ideal. Then R/I is a simple right R-module, so $\forall\, 0 \neq \bar{x} \in R/I$, $\bar{x}a = 0$ by assumption of what a is (specifically $a \in \mathrm{Ann}_R(R/I)$). In particular, $\bar{1}a = 0 \Rightarrow a \in I \Rightarrow a \in J(R)$.

Lemma 7.2 (Nakayama)

The following are equivalent: for a right ideal I,

(i) $I \subseteq J(R)$

(ii) If M is a finitely generated R-module and $N \subseteq M$ satisfying $N + MI = M$, then $N = M$

(iii) $\{1 + x : x \in I\} = G$ is a subgroup of the unit group of R ($R^\times$)

pf: example sheet 4.

Remark (iii) $\Rightarrow$ $J(R)$ is the largest 2-sided ideal J such that $\{1 + x : x \in J\}$ is a subgroup of $R^\times$.

Consequently if we defined $J(R)$ using maximal left ideals, we'd get the same thing.

Dfn 7.3: R is **semisimple** if $J(R) = 0$.

Example: $M_n(D)$ is semisimple.

: $\mathbb{F}_p G$, G cyclic of prime order p, then $\mathbb{F}_p G = \mathbb{F}_p[X]/(X^p - 1)$

$\Rightarrow$ $J(\mathbb{F}_p G) = (X - 1) \bmod (X^p - 1)$.

Lemma 7.4: Let R be a semisimple, finite dimensional (associative) algebra. Then R is the direct sum of finitely many simple (right) R-modules.

proof: $0 = J(R) = \bigcap \{\text{maximal right ideals}\}$

Consider $R \supsetneq I_1 \supsetneq I_1 \cap I_2 \supsetneq \ldots$ where I_j are maximal right ideals. This chain must terminate (fin. dim) so that $0 = J(R) = I_1 \cap I_2 \cap \ldots \cap I_n$, and we may assume n is minimal.

Consider maps
$$R \longrightarrow \bigoplus (R/I_j)$$
$$r \longmapsto (r + I_1, r + I_2, \ldots)$$

$\to$ n is minima

Note $\bigcap_{j \neq i} I_j \neq 0$ by assumption, and the restriction of the map $\theta: R \to R/I_i$ is injective on $\bigcap_{j \neq i} I_j$.
So the image in R/I_i is nonzero, and so it is the whole of R/I_i since R/I_i is simple.

Thus $I_2 \cap \ldots \cap I_n$ corresponds to $(R/I_1, 0, 0, \ldots, 0) \subseteq \bigoplus R/I_j$, and we see that θ is surjective.
Thus θ is an isomorphism as $\ker\theta = \bigcap I_j = 0$.

$\square$

Lemma 7.5: let R be semisimple, M any nonzero, finite dimensional R module, then M is a direct sum of simple modules proof: R semisimple as an algebra $\hookrightarrow$ R semisimple as an R-module. Now $\Rightarrow$ the free R module $\bigoplus_{i=1}^{n} R$ is a completely reducible R-module, where $n = \#$ generators of M. Define a map $F: \bigoplus_{i=1}^{n} R \to M$; $e_{m_i} \mapsto m_i$. Then F is surjective, and $\Rightarrow$ $M \cong {}^{\oplus R}/_{\ker(f)}$. But quotients of completely reducible are completely reducible. So M is completely reducible

(completely reducible)

Definition 7.6: M is completely reducible is it can be written as a direct sum of simple R-modules.

Definition 7.7: The socle $soc(M)$ of a finite dimensional R module M is the sum of all its minimal (nonzero) submodules.

Lemma 7.8: $soc(M) = \{ m \in M : m J(R) = 0 \}$

proof: each minimal submodule M' of M is simple and is $\cong {}^{R}/_{Ann_R(m)}$ for any $m \in M'$, $m \neq 0$.
So $J(R) \leq Ann_R(M') = \bigcap_{m \in M'} Ann_R(m)$. Thus $J(R)$ annihilates M and therefore $soc(M)$.

Conversely, if $m J(R) = 0$ then mR may be regarded as a ${}^{R}/_{J(R)}$ - module. ($\to$ semisimple) So by 7.5, this is semisimple and $\therefore$ a direct sum of simple modules. So $mR \subseteq soc(M)$.
($\hookrightarrow$ minimal.)

each minimal submodule M' is simple, and so M' appears in $J(R) = \bigcap_{M \text{ simple}} Ann_R(M)$
$\quad \cup\!\!\mid$
$Ann_R(M')$
$\Rightarrow J(R) \leq Ann_R(M')$
$\Rightarrow soc(M) \subseteq \{ m \in M : m J(R) = 0 \}$

Definition 7.9: The socle series of M:

$$0 \leq soc_0(M) \lneq soc_1(M) \lneq \qquad \text{if } soc_{i-1}(M) \neq M$$

where
$$soc_i(M) / soc_{i-1}(M) = soc\left(M / soc_{i-1}(M) \right)$$

Remark: 1) The series must terminate at M.
2) $soc_i(M) = \{ m \in M : m J^i = 0 \}$

Proposition 7.10: Let R be a finite dimensional (associative) algebra. Then $J(R)$ is nilpotent (i.e $\exists\, m \in \mathbb{Z}_{>0}$ s.t $J^m = 0$).

proof: let $J = J(R)$. Consider $R \gneq J \gneq J^2 \gneq J^3 \gneq \cdots$. This must terminate, $J^n = J^{n+1}$ for some n.
So the socle series must terminate, so $R = soc_n(R)$ for n. Then J^n annihilates 1, and so $J^n = 0$.

Now consider the semisimple quotient ${}^{R}/_{J(R)}$. Set $J(R) = 0$ and consider the endomorphisms of R_R (R as a right R-module).

Lemma 7.11: (Schur's Lemma)
Let S be a simple right R-module. Then $End_R(S)$ is a division ring. If S_1 and S_2 are non-isomorphic simple R-modules, then $Hom_R(S_1, S_2) = \{0\}$.

Note: S is a left $End_R(S)$ - module.

proof: Let $\phi: S \to S$ be an R-module homomorphism. Then either $\phi(S) = 0$ i.e. $\phi = 0$, or $\phi(S) = S$ using simplicity of S. Furthermore, $\ker\phi$ is a submodule of S and so either $\ker\phi = 0$ or $\ker\phi = S$. So if $\phi \neq 0$, then ϕ is bijective and has a right and left inverse. Thus $\mathrm{End}_R(S)$ is a division ring.

If $S_1 \not\cong S_2$ and $\phi: S_1 \to S_2$ with $\phi \neq 0$ then $\ker\phi = 0$, $\mathrm{Im}\phi = S_2$ and ϕ is an isomorphism $\lightning$.

Lemma 7.12: Regarding R as a right R-module (R_R), then $\mathrm{End}(R_R) \cong R$ via multiplication on the left by elements of R.

proof: $\phi \in \mathrm{End}_R(R)$, then ϕ is determined by $\phi(1)$. The map $\mathrm{End}(R_R) \to R$; $\phi \longmapsto \phi(1)$ is an isomorphism, noting that multiplication by $\phi(1)$ on the left is the endomorphism ϕ.

Theorem 7.13: (Artin-Wedderburn) Let R be a semisimple finite dimensional associative algebra over a field K. Then $R = \bigoplus_{i=1}^{r} R_i$, where $R_i = M_{n_i}(D_i)$ for a finite dimensional (division) algebra D_i, and the R_i are uniquely determined. R has exactly r isomorphism classes of right simple modules S_i and $D_i = \mathrm{End}_R(S_i)$, and $\dim_{D_i}(S_i) = n_i$.

Furthermore, if K is algebraically closed then $D_i = K$ $\forall i$.

Remark: $\mathbb{C}G$ is semisimple for a finite group, and so the theorem says that $\mathbb{C}G$ is the direct sum of matrix algebras over $\mathbb{C}$, where the number of matrix algebras is equal to the number of simple modules up to iso.

Corollary 7.14: if G is a finite group, $Z(\mathbb{C}G)$ is an r-dimensional $\mathbb{C}$-vector space, and $r = \#$ of isomorphism classes of simple modules $= \#$ of conjugacy classes.

proof: any class sum $\sum_{g' \in cl(g)} g' \in Z(\mathbb{C}G)$. Any element of $Z(\mathbb{C}G)$ must be a linear combination of class sums.

The class sums for the various conjugacy classes are a basis of $Z(\mathbb{C}G)$. So $\dim(Z(\mathbb{C}G)) = \#$ of conj. classes in G.

But AW $\Rightarrow$ $Z(\mathbb{C}G) = \bigoplus_{i=1}^{r} Z(M_n \mathbb{C})$, and $Z(M_n\mathbb{C}) = \{\text{set of scalar matrices}\}$

So $\dim Z(\mathbb{C}G) = \#$ of direct summands $= \#$ of isomorphism classes. $\square$

proof of AW)

(7.5) $\Rightarrow$ ${}^{R}R$ is a finite direct sum of simple right modules. Group those that are isomorphic to eachother. Then ${}^{R}R = (S_{11} \oplus \cdots S_{1n_1}) \oplus (S_{21} \oplus \cdots \oplus S_{2n_2}) \oplus \cdots (\quad)$, so that $S_{ik} \cong S_{il}$ ($\cong S_i$, say), but $S_{ik} \not\cong S_{jl}$ if $i \neq j$.
Let $R_i := S_{i1} \oplus \cdots S_{in_i}$. Thus $R = \oplus R_i$. Now let S be a simple R-submodule of R_R. Consider projections $\pi_{ik} : R \to S_{ik}$ restricted to S. By Schur's Lemma, $\pi_{ik}|_S$ is either zero or an isomorphism. Note that at least one of these restrictions must be nonzero. So $\pi_{ik}|_S$ is nonzero for exactly one i (and possibly various k), and thus S has to lie in R_i. Hence we deduce that R_i is the sum of all simple submodules of R_R which are isomorphic to S_i, and $\therefore$ is uniquely determined (recall $S_{il} \cong S_i \ \forall l$)

Consider $\mathrm{End}(R_i) = \mathrm{End}_R(S_{i1} \oplus \cdots \oplus S_{in_i}) \cong M_n(D_i)$, where $D_i = \mathrm{End}_R(S_i)$ by (7.11) Schur. In particular, Schur says that D_i is a division algebra

Remark: $\phi \in \mathrm{End}_R(S_{i1} \oplus \cdots \oplus S_{in_i})$ is represented by a matrix (ϕ_{ml}) where $\phi_{ml} \in \mathrm{Hom}(S_{il}, S_{in})$. However, $R = \mathrm{End}_R(R_R)$ by (7.12), so

$$R \cong \begin{pmatrix} M_{n_1}(D_1) & 0 & 0 & \cdots \\ 0 & M_{n_2}(D_2) & 0 & \cdots \\ 0 & 0 & \ddots & 0 \end{pmatrix}$$

With zero blocks since $\mathrm{Hom}(S_{in}, S_{jl}) = 0$ if $i \neq j$.

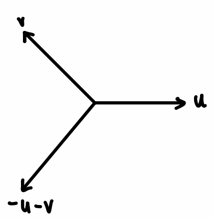
It is left to show that $\dim_{D_i}(S_i) = n_i$. Consider $M_{n_i}(D_i)$. It breaks up as a direct summand of simple right modules $\left(\frac{}{0}\right), \left(\frac{0\ 0\ \cdot}{0}\right), \ldots$. Each of these gives a simple submodule $\cong$ {row n_i-tuples}
Hence $\dim_{D_i}(S_i) = n_i$.

zero apart from one row

$\square$

Example: $G = S_3$, k an algebraically closed field. Let g a transposition and h a 3-cycle. In char $k = 0$, $\exists$ 3 simple modules up to isomorphism (3 conjugacy classes).

let U_1 = trivial, 1 dimensional module, where g, h act trivially
U_2 = g transposition acts like -1, h acts like $+1$. (1 dim)
$U_3 = K^2$ with g acting via $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. 2 dimensional representation.



representing wrt basis $\{u, v\}$ (right module first row tells image of first basis vector)

g: $u \mapsto -u-v$
$v \mapsto v$

h. $u \mapsto -u-v$
$v \mapsto u$

In characteristic 2: $\bar{u}_1 = \bar{u}_2 \mod 2$

$\bar{u}_3$ is still simple, 2 dimensional.

This gives 2-simple modules. So Artin-Wedderburn $\Rightarrow$ $^{KG}/_{J(KG)} \cong$ direct sum of matrix algebras $M_{n_i}(K)$.
Where $n_i := \dim_K$ of the corresponding simple modules.

So $^{KG}/_{J(KG)} \cong \underbrace{M_1(K) \oplus M_2(K)}_{5 \text{ dimensional}} (\oplus \underset{?}{\cdots})$, $\dim_K(KG) = 6$.

However, $\gamma = 1 + h + h^2 + g + gh + gh^2$ = sum of all elements. This is central in KG, and $\gamma^2 = 0$ in KG since char $K = 2$. $(\gamma^2 = 6\gamma = 0)$. But γ central $\Rightarrow$ γKG is 2-sided ideal $\Rightarrow$ nilpotent, so $\gamma KG \subseteq J(KG)$.
So $\dim(J(KG)) \geq 1$ $\Rightarrow$ $^{KG}/_{J(KG)} = M_1(K) \oplus M_2(K)$.
$\hookrightarrow J(K,G) = \gamma KG$.

And $\mathrm{soc}(KG) = \{x \in KG : xJ(KG) = 0\} = \{x : x\gamma = 0\}$. Notice $g-1 \in \mathrm{Soc}(KG)$, and $h-1 \in \mathrm{Soc}(KG)$. Hence $\mathrm{Soc}(KG) = \ker(KG \longrightarrow K)$; gp element $\longmapsto 1$, which is an ideal of codimension 1.

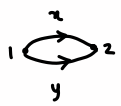
Exercise: do same for char 3: $\bar{u}_1, \bar{u}_2$ are non-isomorphic = 2 simple modules, but $\bar{u}_3$ is no longer simple.
Find $\dim(J) = 4$, and $^{KG}/_{J(KG)} \cong M_1(K) \oplus M_1(K)$.

8 QUIVERS

Definition 8.1: A **quiver** Q is a directed (multi)-graph, with vertices labelled by $\{i\}$ and arrows $i \longrightarrow j$.
There is no restriction on # of arrows between i and j. We also allow loops $\circlearrowleft$

Definition 8.2: A **representation M of Q** is a direct sum of vector spaces M_i, $\oplus M_i$, where i is the label of vertices, together with linear maps $\theta_x : M_i \to M_j$ for each arrow $i \xrightarrow{x} j$

Example:



$M_1 = k$ $\quad$ θ_x, θ_y are zero maps

$M_2 = 0$

Definition 8.3: A **morphism of representations** is a collection of linear maps $M_i \to M_i'$ which commute with the linear maps representing the edges.

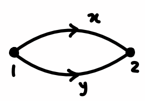
Definition 8.4: A **path of length $\ell \geq 1$** is a concatenation of ℓ compatible arrows. For an arrow $i \longrightarrow j$ can define its source as i and target as j. Two arrows are compatible if the target of one is the source of the other

e.g. $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ path of length 3

A path of length 0 is just a vertex.

Definition 8.5: The **path algebra** kQ is a k-v.s. with basis given by the paths, and the multiplication is given by concatenation of compatible paths. If two paths are incompatible, then their product is zero.

Example:



paths of length $0 = \{e_1, e_2\}$

$1 = \{x, y\}$

$\geq 2 = \emptyset$

products: $e_1 x = x$, $e_1 y = y$, $e_2 x = 0$, $e_2 y = 0$

$x e_1 = 0$, $y e_1 = 0$, $x e_2 = x$, $y e_2 = y$

$xy = yx = 0$ $\qquad$ $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = e_2 e_1 = 0$

Note: paths of length 0 (corresponding to vertices) give idempotents.

Lemma 8.6:
a) kQ is finite dimensional $\Leftrightarrow$ Q is finite and it contains no directed cycles
b) If Q is finite, then kQ is finitely generated.

pf: a) kQ fin. dim $\Leftrightarrow$ $\exists$ only finitely many paths.

b) note that kQ is generated by $\{e_i\}$ corresponding to vertices and $\{x\}$ corresponding to edges.

In fact the converse of b) is also true. $\square$

Suppose M is a representation of our quiver Q: $M = \oplus M_i$ and if $\exists$ an edge $i \xrightarrow{x} j$, then x acts on M_i by applying θ_x.

Thus $\oplus M_i$ can be thought of as a kQ-module. We get a correspondence

$$\{kQ\text{-modules}\} \longleftrightarrow \{\text{representations of } Q\}$$

Note that there are simple modules $S_i \longleftrightarrow$ representation $:= \begin{cases} k & \text{at vertex } i \\ 0 & \text{otherwise} \end{cases}$ + all maps are zero.

They are non isomorphic.

> Idea: Generators of kQ are vertices $\{e_i\}$ and edges. Note that $e_j e_i = \delta_{ij}$, and $x e_i = x$ if source of x is e_i, and 0 otherwise (writing composition reading right to left, to reflect their action as maps). The identity of kQ is $e_1 + \dots + e_n$. By this $e_j e_i = \delta_{ij}$ fact, we have that V decomposes as the sum
>
> $$V = \bigoplus_{i=1}^{n} \underbrace{e_i V}_{V_i}$$
>
> And from this we can recover the action of the arrows / edges. Say $i \bullet \xrightarrow{x} \bullet j$, then we can write x as $e_j x e_i$. Then $e_j x e_i (V_i) = e_j x e_i e_i V = e_j x V \subseteq e_j V$, so we can think of x as a linear map $V_i \to V_j$.
>
> We can gather what simple modules would look like. If we had say V_i and V_j both nonzero for $i \neq j$, then certainly we would have a submodule e.g. V_j of $V_i \oplus V_j \oplus (\dots)$. So only have one nontrivial V_i, and certainly rest of $V_j = 0 \Rightarrow$ all maps are 0. For V_i to be a simple kQ-module, $\Rightarrow V_i = k$.
>
> The fact that they're all distinct (nonisomorphic): if S and S', say $S = S_i$ and $S' = S_j$, then $V_i \cong V_i'$ since this iso must respect the decomposition $\Rightarrow S_i = S_j$, but $i \neq j$ so $\lightning$.

Example:

$$\begin{tikzcd} 1 \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & 2 \end{tikzcd}$$

$e_1 kQ \longleftrightarrow$ repn $M_1 = k$, $M_2 = k \oplus k$, $\theta_x : M_1 \to M_2$; $\lambda \mapsto (\lambda, 0)$ $\quad \theta_y : M_1 \to M_2$ $\lambda \mapsto (0, \lambda)$

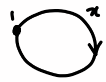
Example: Q finite, no directed cycles, and simple modules as described before previous example. Then these S_i are the only simple modules of kQ.

To see this, consider $J := \bigcap_i \mathrm{Ann}(S_i) = k$-span of paths of length ≥ 1

then $J^r = k$-span of paths of length $\geq r$. Hence $\exists\, n \in \mathbb{N}$ s.t. $J^n = 0 \Rightarrow J$ is a nilpotent ideal.
So $J \subseteq J(kQ)$ since J is nilpotent. Clearly $J(kQ) \subseteq J$ from the definition of $J(kQ)$, and hence $J = J(kQ)$.

Note $kQ/J \cong \overbrace{k \oplus \dots \oplus k}^{\text{corresponding to vertices.}}$. But also $kQ/J = kQ/J(kQ)$ which is semisimple, so Artin-Wedderburn says that the decomp kQ/J is unique and # of summand uniquely determined, corresponding to the iso classes of simple modules. As the sum has # vertices summands, and we already have # vertices distinct simple kQ-modules S_i, these (up to iso) must be all of them.

Definition 8.7: An algebra R has **finite representation type** if there are only finitely many indecomposable modules (up to isomorphism).

Example Q: 

Representation: $M_1 = K$, $\theta_x : M_1 \to M_1$; $\lambda \mapsto \lambda\mu$ for a fixed $\mu \in K$

This is clearly indecomposable (1-dimensional), and are non isomorphic to those for different μ.

So if K is infinite, then kQ does not have finite representation type (infinitely many reptns).

(Exercise: remove K-finite restriction)

(Exercise: Show that if Q contains a directed cycle, then $\exists$ infinitely many indecomposable modules (whether K is infinite or not). Similar construction.

(Exercise: Q  . Show also has infinitely many indecomposable representations up to iso).

algebraically closed fields are infinit

underlying graph

Theorem 8.8 (Gabriel, 1972): let k be an algebraically closed field. A connected quiver has a path algebra of finite representation type if and only if its underlying graph (ignoring directions) is of type A_r $(r \geq 1)$, D_r $(r \geq 4)$, E_6, E_7, E_8 (the simply laced Coexter graphs).

Remarks (1) this is independent of the direction of the arrows.

(2) If we drop the algebraically closed restriction, we can get other coexter graphs, e.g. B_r, C_r, F_4, G_2.

(3) the more general theorem is a classification of <u>positive-definite</u> Coexeter graphs.

Given a Coexter graph, we can define a symmetric bilinear form on the $\mathbb{R}$-span of the vertices $v_1, \dots, v_n$ (say), which form a basis for this vector space.

$$q_{ij} = \begin{cases} 2 & \text{if } i=j \\ -\sqrt{t_{ij}} & \text{if } i \neq j \end{cases}$$

no direction in coexeter graph

where t_{ij} = # of edges connecting the two vertices

If a Coexeter graph arises from a root system, say $\Delta = \{\alpha_1, \dots, \alpha_r\}$ of root system Φ, then

$$q_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_i| \, |\alpha_j|}$$

Symmetrized version of the cartan matrix

Recall that root system Φ gives coxeter graph with vertices the simple roots, and # of edges between α and β given by $n(\alpha, \beta)\, n(\beta, \alpha)$

Recall: $n(\beta, \alpha) = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$

idk.

$$\Rightarrow n(\alpha, \beta)\, n(\beta, \alpha) = \frac{4(\alpha, \beta)(\beta, \alpha)}{(\alpha, \alpha)(\beta, \beta)} = \frac{(2(\alpha, \beta))^2}{|\alpha|^2 |\beta|^2} = \left(\frac{2(\alpha, \beta)}{|\alpha| |\beta|}\right)^2$$

Note that this matrix is the same as the one representing the inner product wrt basis $\{\frac{\alpha_i}{|\alpha_i|}, \alpha_i \in \Delta\}$.
This matrix is therefore positive definite ! Semisimple Lie algebras give positive definite coxeter graphs!

Definition 8.8: A coexeter graph is positive definite if the symmetric bilinear form defined by the q_{ij} is itself positive definite.

Lemma 8.9: A connected positive definite coexeter graph with r vertices has that the number of pairs of vertices joined by at least one edge $= r-1$.

proof: Let $e = \#$ of pairs of vertices joined by at least one edge. Let $v = \sum_{i=1}^{r} v_i$. Then $v \neq 0$, since basis, and so

$$0 < q(v,v) = 2r + 2\sum_{i<j} q_{ij}$$

But for i,j distinct, $q_{ij} \leq 0$, and so $r > -\sum_{i<j} q_{ij} \overset{(*)}{=} \sum_{i<j} \sqrt{t_{ij}} \geq e \} \Rightarrow e \leq r-1$

has to be at least 1

But the graph is connected, and so we must have $e \geq r-1 \Rightarrow e = r-1$.

Definition 8.10: The dimension vector of a representation.

$$\sum (\dim M_i) v_i \in \mathbb{R}^r \qquad i = \text{vertices}, \quad v_1, \dots, v_r \text{ basis} \subseteq \mathbb{R}^r$$

Theorem 8.11 (Gabriel) Suppose the underlying coexeter graph of a quiver Q is a simply laced Coexeter graph of type A_r, D_r, E_6, E_7, E_8. Then

the isomorphism classes of indecomposable representations $\longleftrightarrow$ positive roots in $\mathbb{R}^r$ wrt. $\Delta = \{\alpha_1, \dots, \alpha_r\}$

$$M \longrightarrow \sum \overset{\dim(M_i)}{k_i} v_i \longleftrightarrow \underbrace{\sum k_i \overset{\text{simple roots.}}{\alpha_i}}_{\text{want this to be +ve root.}}$$

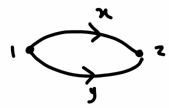
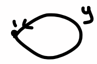
Proof of 8.8: It makes use of some reductions:

1) Given a quiver Q, remove some vertices and any arrow with source or target among the removed vertices, to give quiver Q'.

Then if Q' has infinitely many indecomposable representations (up to iso), then Q does too.
We could put the subspace 0 at any of the removed vertices, and the zero map representing any removed arrow.

Then Q has fin. rep. type $\Rightarrow$ Q' does too.

2) Given Q, contract along an arrow and identify the source and target of the arrow to get Q'

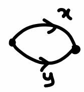
e.g.  contract along x : 

Given a representation of Q' we can form a representation of Q by putting the same vector space at the source and target of the contracted edge and represent the contracted edge by the identity map: So Q fin. rep type $\Rightarrow$ Q' does too.

e.g. $V_1 = V_2 = K$

$\theta_x : \lambda \longmapsto \lambda\mu$ $\quad \mu \in K$ (infinitely many)

$\theta_y : \lambda \longmapsto$

e.g.  has infinite rep. type $\Rightarrow$  does too.

We can use these two sorts of reduction to deduce that the underlying graph of a quiver of finite representation type has to be a tree (without multiple edges).

Now assume Γ, the underlying graph of Q, is a tree. As before, we define a symmetric bilinear form on $\mathbb{R}^r$ with basis $v_1, \dots, v_r$ (corresponding to vertices $1, \dots, r$). We defined wrt. basis

$$q_{ij} = \begin{cases} 2 & i = j \\ -1 & \text{if } i \text{ and } j \text{ are adjacent in } \Gamma \end{cases}$$

going to show that if we have a quiver w/ fin rep^n type, then bilinear form must be +ve def.

Suppose this symmetric bilinear form is not positive definite for contradiction's sake. Then $\exists$ nonnegative integers k_i s.t $q(\underline{v}, \underline{v}) \leq 0$ with $v = \sum k_i v_i \neq 0$.

on underlying graph

Evaluating $q(\underline{v}, \underline{v})$ gives $2\sum k_i^2 - 2\sum\limits_{i \leftrightarrow j} k_i k_j$. So $q(\underline{v}, \underline{v}) \leq 0 \Rightarrow 2\sum k_i^2 \leq \sum\limits_{i \leftrightarrow j} k_i k_j$

Thus $2\sum k_i^2 \leq 2\sum k_i k_j$ where i, j are adjacent in Γ. So

not double counting in sum

$$\sum k_i^2 \leq \sum_{\substack{i \to j \\ \text{in } Q}} k_i k_j \qquad (*)$$

Stops us double counting.

Let M_i be a vector space of dimension k_i, and $M = \oplus M_i$. So $v = \sum k_i v_i$ is the dimension vector of M. Define a linear map $M_i \to M_j$ to represent each arrow $i \longrightarrow j$. Consider isomorphism classes of such representations. Two representations are isomorphic $\Leftrightarrow$ there is an automorphism in $\prod\limits_i GL(M_i)$ taking one to the other. Then if $f_i \in GL(M_i)$,

$$\begin{tikzcd} M_i \arrow[r, "\theta"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M_i \arrow[r, "f_j \circ \theta \circ f_i^{-1}"'] & M_j \end{tikzcd}$$

want this to commute so this bottom map better be $f_j \circ \theta \circ f_i^{-1}$

So we consider the orbits of $\prod GL(M_i)$ on $\prod\limits_{i \to j} \mathrm{Hom}(M_i, M_j)$. But $\prod\limits_i GL(M_i)$ is an algebraic variety of dimension $\sum k_i^2$.

acting as in the square

$\prod \theta$'s in here

these are scalars

$\begin{cases} f_i = \lambda I \text{ for } I \in GL(M_i) \\ \text{the identity, same } \lambda \\ \text{for all } I \end{cases}$

Similarly $\prod \mathrm{Hom}(M_i, M_j)$ is an algebraic variety of dimension $\sum_{i\to j} k_i k_j$. Moreover the scalars in $\prod GL(M_i)$ act trivially on $\prod \mathrm{Hom}(M_i, M_j)$ and so actually have an action of $\prod GL(M_i)/$ scalars with dimension $\sum k_i^2 - 1$.

↳ look at commuting square + previous blue comment

By $(*)$ this dimension $\lneq$ $\dim \prod_{i\to j} \mathrm{Hom}(M_i, M_j)$. remember $(*)$: $\sum (k_i)^2 \leq \sum_{i\to j} k_i k_j \Rightarrow \sum (k_i)^2 - 1 < \sum_{i \to j} k_i k_j$

So we must have infinitely many orbits, and so we have infinitely many repns (up to iso) with this dimension vector $\sum k_i v_i$. And $\dim(M) = \sum k_i = \ell$, say. But if Q is of fin. rep type, there are only finitely many isomorphism classes of representations of dimension ℓ ↯. So Q cannot have fin. rep. type.

↳ because there's only finitely many indecomposable ones. → all fin. dim. representations are finite sums (perhaps trivial) of indecomposable representations.

We've shown that Γ (underlying graph of a quiver of finite repn type) has to be a positive definite Coexeter graph without multiple edges (so simply laced). Our classification of semisimple Lie Algebras of type A_r, B_r, C_r, D_r and E_6, E_7, E_8, F_4 and G_2 was actually arising from the classification of positive definite Coexeter graphs.

Thus, for Q of finite rep. type, we are restricting to the simply laced graphs on our list.

> An aside: Suppose we have a coxeter graph C that arises from a root system Φ. Then C is connected iff Φ is irreducible. Also, Φ is simply laced $\Leftrightarrow$ C is a tree
>
> If reducible then obviously disjoint. If disjoint, can partition base $\Delta = \Delta' \sqcup \Delta''$, where Δ' from one component and Δ'' from the other. Let $\Phi' = W\Delta'$ and $\Phi'' = W\Delta''$. Then $\Phi = \Phi' \cup \Phi''$. To see this is actually disjoint, note that by table $n(\alpha,\beta) = n(\beta,\alpha) = 0$ for $\alpha \in \Delta'$ and $\beta \in \Delta''$, so $(\alpha,\beta) = 0$. But $(\cdot,\cdot)$ is invariant under W since W is generated by reflections (inner prod. preserving). So $\forall \alpha \in \Phi'$, $\beta \in \Phi''$, $(\alpha,\beta) = 0$ so we have actually $\Phi = \Phi' \sqcup \Phi''$, i.e. Φ reducible. Simply laced $\Leftrightarrow$ tree comes from table.

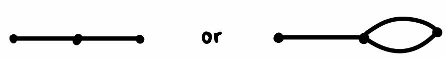
The strategy for proving the classification of positive definite coexeter graphs (that they're of type A_r, B_r, C_r, D_r, F_4, G_2, E_6, E_7, E_8) is similar to what we were doing earlier to get down to a tree.

reductions

think about small cases first.

Step 1: The only connected positive definite Coexeter graphs with 3 vertices are

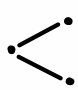
lemma 8.9 + pf of 8.8.



2 vertices → triple edges are as far as you can go

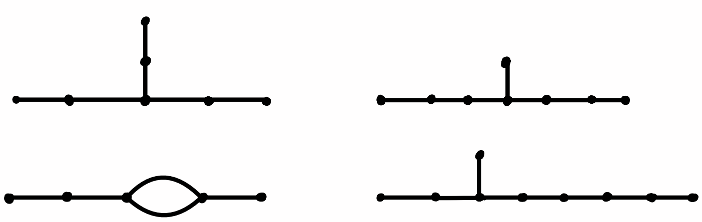
Step 2: if Q is a positive definite quiver with an edge, then we can contract the edge to give a positive definite Coexeter graph.

contract green edge

Similarly if  is part of a valid graph, then I can replace it by 

Step 3: can't have:

rule out particular graphs



remember: simply laced iff tree. So not simply laced guys are B_r, F_4 and G_2.

Step 4: Show we're down to a list $A_r, B_r, D_r, E_6, E_7, E_8, F_4, G_2$.

Alternative proof: use representations throughout (Pierce's book)

same as 8.11 but numbering messed up at 8.8

Theorem 8.12 (Gabriel) Q quiver, underlying Coexeter graph which is simply laced + positive definite. i.e. A_r, D_r, E_6, E_7, E_8). Then

↳ arising from some root system.

$$\left\{\begin{array}{c}\text{Isomorphism classes of finite} \\ \text{dimensional indecomposable} \\ \text{representations of } Q.\end{array}\right\} \longleftrightarrow \{\text{positive roots in } \mathbb{R}^r\}$$

the isomorphism classes of indecomposable representations $\longleftrightarrow$ positive roots in $\mathbb{R}^r$ wrt. $\Delta = \{\alpha_1, \dots, \alpha_r\}$

$$M \longrightarrow \Sigma \overset{\dim(m_i)}{k_i} v_i \longleftrightarrow \underbrace{\Sigma k_i \overset{\text{simple roots.}}{\alpha_i}}_{\text{want this to be +ve root.}}$$

dimension vectors $\Sigma k_i v_i \longleftrightarrow \Sigma k_i \alpha_i$ $\qquad$ α_i simple roots. Thus kQ has finite representation type.

remember we can think of such a rep^n as a kQ module via action of vertices and edges extended.

Then we know since $\exists$ finitely many +ve roots, there are finitely many iso classes of fin. dim. indecomp. reps. of Q. I.e. kQ has fin. rep^n type.

First of all we need to think about numberings of vertices. We want to use a well chosen one dependent on the direction of the arrows in Q.

Definition 8.13 A vertex of Q is a sink if all the arrows meeting it have the vertex as the target (not start of any arrows).

A vertex is a source if all the arrows meeting it have the vertex as the source.

Clearly any finite quiver without directed cycles has sources and sinks.

definitely has, but not necessarily every vertex either a sink or source.

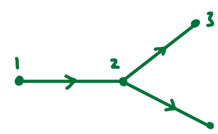
Definition 8.14 Given a quiver Q, define a new quiver $s_i Q$ with the same vertices but with the orientation of the arrows meeting vertex i reversed.

Example Q $\qquad \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$

$s_2 Q$ $\qquad \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet}$

$s_1 Q$ $\qquad \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$

e.g



clearly need $1 < 2$, and $2 < 3, 4$

1 a source, and 4 a sink (3 also a sink but whatever)

We can number the vertices $1, \dots, r$ so that $i < j$ for any arrow which goes $i \to j$. And so vertex 1 is a source, and vertex r is a sink.

Definition 8.15: such a numbered quiver is a standardised quiver.

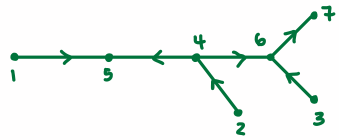
Lemma 8.16: For a standardised quiver Q

1) if $1 \le j < r$, then j is a sink and $j+1$ is a source of the quiver $s_j s_{j-1} \cdots s_2 s_1 Q$

2) if $1 < j \le r$, then j is a source and $j-1$ is a sink of the quiver $s_j s_{j+1} \cdots s_r Q$

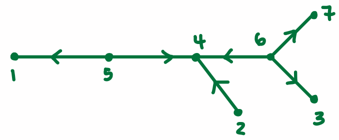
3) if $s_1 s_2 \cdots s_r Q = s_r s_{r-1} \cdots s_1 Q = Q$

Example: Q :



Take $j = 4$. Then

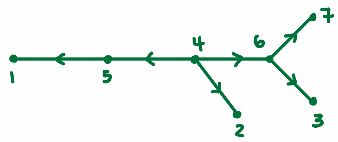
(1) $s_4 s_3 s_2 s_1 Q$:



Then we see that 4 becomes a sink and $4+1 = 5$ becomes a source.

And

(2) $s_4 s_5 s_6 s_7 Q$:



Now 4 is a source and $4-1=3$ is a sink.

Finally: can run through $s_7 s_6 s_5 s_4 s_3 s_2 s_1 Q = s_1 s_2 s_3 s_4 s_5 s_6 s_7 Q = Q$

we have an arrow

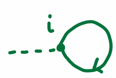
proof: follows from Q being standardised, and that if $j_1, \dots, j_s$ are distinct vertices then $i \longrightarrow j$ in $s_{j_1} s_{j_2} \cdots s_{j_s} Q$
if either • $i \longrightarrow j$ in Q and either none or both of i and j appear among the $j_1, \dots, j_s$
or • $i \longleftarrow j$ in Q and exactly one of the i and j appear among the $j_1, \dots, j_s$

For 3) note that both $s_1, \dots, s_r$ and $s_r, \dots, s_1$ have reversed orientation of each arrow twice: $i \to j$ reversed by s_i and s_j

Definition 8.17: numbering of vertices **is admissible** if for each j, j is a sink of $s_{j+1} s_{j+2} \cdots s_r Q$

Lemma 8.18: There is an admissible numbering for the vertices of Q iff has no directed cycles.

proof: 8.16 tells us that a standardised quiver has an admissible numbering. Clearly if we do have a directed cycle there isn't an admissible numbering.

↳ if Q has a directed cycle, say  , then i is simultaneously both a target and a source, so can never be a sink or a source ($\forall$ arrows with i an end). Same goes for any directed cycle.

The first part follows from lemma 8.16 part 2, which says that j is a source of $s_j \cdots s_r Q$ $\forall\ 1 < j \leq r$, and so $s_{j+1} \cdots s_r Q$ must be a sink $\forall$ $1 < j \leq r$. And certainly when $j = 1$, $s_1 \cdots s_r Q = Q$, and 1 is a source of Q, so 1 is a sink of $s_2 \cdots s_r Q$.

Exercise: given two quivers Q and Q' with the same underlying graph which is a tree, then there is some choice of $j_1, \dots, j_s$ such that $s_{j_1} \cdots s_{j_s} Q = Q'$.

Now suppose j is a sink of Q.

Definition 8.19 We define functors $S_j^+ :$ Q-representations $\longrightarrow$ s_jQ-representations

$S_j^- :$ s_jQ-representations $\longrightarrow$ Q-representations

Given a representation of Q, V, let $S_j^+(V) = W$ where $W_i = V_i$ for $i \neq j$ and $W_j =$ kernel of $\phi = \oplus$ of maps representing the arrows with target j

same k-vector spaces for each vertex $i \neq j$, but we need maps in reverse direction and so we need to change V_j

We have a map $0 \longrightarrow W_j \xrightarrow{\text{inc}} \bigoplus_{\substack{i \text{ s.t.} \\ i \to j}} V_i \xrightarrow{\phi} V_j$ (†) $W_j = \ker(\phi)$

Picture to have in mind: say we have

Q: $i \xrightarrow{\varphi} j \xleftarrow{\gamma} k$ then s_jQ = $i \longleftarrow j \longrightarrow k$

If $\{V_i, V_j, V_k, \varphi : V_i \to V_j, \gamma : V_k \to V_j\}$ a repⁿ of Q, then take V_i, V_k, and $W_k = \ker\varphi \oplus \ker\gamma \subseteq V_i \oplus V_k$, so we get natural proj. maps $pr_i : W_j \subseteq V_i \oplus V_k \to V_i$ and $pr_k : W_j \subseteq V_i \oplus V_k \to V_k$.

Note that there are obvious maps $W_j \to V_i = W_i$ for each i where $i \to j$ in Q given by projection (W_j is a subspace of $\bigoplus_{i \text{ s.t. } i \to j} V_i$) and therefore for each i s.t. $j \to i$ in s_jQ.

So $W = \oplus W_i$ is a representation of s_jQ (for other arrows we represent by the same maps as before for Q).

The functor S_j^- is the dual of this. Given a representation W of s_jQ, let $V_i = W_i$ if $i \neq j$. Set V_j = cokernel of the Σ of the maps representing arrows with source in s_jQ.

We have a map $W_j \xrightarrow{\psi} \bigoplus_{\substack{j \to i \\ \text{in } s_jQ}} W_i \longrightarrow V_j \to 0$ (††)

don't get confused: W repⁿ of s_jQ
V repⁿ of Q !

Again, picture to have in mind: now s_jQ has j a source:

s_jQ: $i \xleftarrow{\varphi} j \xrightarrow{\gamma} k$ Q: $i \longrightarrow j \longleftarrow k$

Say we have repⁿ $\{V_i, V_j, V_k, \varphi : V_j \to V_i, \gamma : V_j \to V_k\}$. Then for Q we can take V_i, V_k as before, but for j now we take $W_j := \Sigma$ cokernel of maps going out of V_j = $\operatorname{coker}(\varphi) \oplus \operatorname{coker}(\gamma) \subseteq V_i \oplus V_k$, and so our corresponding homomorphisms are I guess maybe restriction to the cokernel of each component & including into direct sum? I guess maybe.

If V is a representation of Q for which ϕ in (†) is surjective, then $S_j^- S_j^+(V) = V$, so S_j^+ and S_j^- give a categorical equivalence

$$\left\{ \begin{array}{c} \text{repns of } Q \text{ for} \\ \text{which } \phi \text{ surjective} \\ \text{in (†)} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{repns of } s_jQ \text{ for} \\ \text{which } \psi \text{ injective} \\ \text{in (††)} \end{array} \right\}$$

Now consider indecomposable representations V of Q. Either ϕ is surjective in (†), or V is the irreducible 1-dim representation with $V_j = K$ and $V_i = 0$ otherwise. This follows since if ϕ is not surjective, then we have a splitting $V = V' \oplus V''$ where ϕ is surjective in V' and $V'' = \text{coker}\,\phi$, which gives us a repⁿ with cokerϕ at vertex j and 0 at all other vertices. So V is a decomposable repⁿ ↯. The only exception to this is the case where V is a simple KQ-module, corresponding to $V_j = K$ and 0 everywhere else.

So we've shown

8.20 Lemma: s_j^- and s_j^+ give a bijection between

$$\left\{ \begin{array}{l} \text{indecomposable rep}^{\text{n}}\text{s} \\ \text{of } Q \neq \text{irreducible} \\ \text{rep}^{\text{n}} \text{ of dimension } 1 \\ \text{concentrated at vertex } j \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Indecomposable rep}^{\text{n}}\text{s} \\ \text{of } s_jQ \neq \text{irreducible} \\ 1\text{-dim rep}^{\text{n}} \text{ concentrated} \\ \text{at vertex } j \end{array} \right\}$$

8.21 Corollary: KQ has finite repⁿ type (has finitely many isomorphism classes of indecomposable KQ-modules) if and only if Ks_jQ has finite repⁿ type

Remark: Combining with exercise , we see that if Q and Q' have the same underlying graph which is a tree, then can get from Q to Q' by applying some sequence of s_j's, and so whether Q has finite repⁿ type depends only on the underlying graph of Q (Either both Q and Q' have fin. repⁿ type, or neither have fin repⁿ type).

Now we consider dimension vectors. Given V repⁿ of Q with ϕ surjective, then

$$\dim W_j = \sum_{\substack{i \to j \\ \text{in } Q}} \dim V_i - \dim(V_j) \quad \text{by rank nullity.}$$

$$\dim W_i = \dim V_i \quad i \neq j.$$

Then the effect of applying S_j^+ sends dim vector of V to s_{α_j}(dim vector of V), where s_{α_j} is reflection wrt simple root α_j corresponding to the vertex j, using the fact that our root system is simply laced ($n(\alpha,\beta) = n(\beta,\alpha) = \pm 1$ and in particular for a reduced system we have $n(\alpha,\beta) = -1$, so $s_\alpha(\beta) = \beta - n(\beta,\alpha)\alpha = \beta + \alpha$. Then I think you just run through calc and regroup.

Definition 8.22: A **Coxeter element c** of the Weyl group $\Phi(\Phi)$ is one which is a product of each simple reflections exactly once, in any order.

The Coxeter elements are not unique, but they are all conjugate, and hence of the same order, called the **Coxeter number h of $W(\Phi)$**.

In particular: $h = \dfrac{\#\text{ of roots}}{\text{rank } r}$, and so the dimension of the Lie algebra associated with Φ = # of roots + r = $hr + r = (h+1)r$.

dimension of space

See lemma 4.15 and dfn 4.17: $\dim(L) = \overset{r}{\overset{\shortparallel}{\dim(H)}} + \sum_{\alpha\in\Phi} \overset{1}{\overset{\shortparallel}{\dim(L_\alpha)}} = r + \#\text{ of roots}$

where we recall that the α_i span the dual space H^*, thats our Euclidean space with $(\alpha,\beta) = \langle h_\alpha, h_\beta \rangle_{ad}$, $h_\alpha, h_\beta \in H$

using duality isomorphism w/ non-deg. of $\langle \cdot,\cdot \rangle_{ad}$ restricted to H.

Example:

Φ type	Coxeter number
A_r	$r+1$
B_r	$2r$
C_r	
D_r	$2r-2$
E_6	12
E_7	18
E_8	30
F_4	12
G_2	6

— Coxeter element is $(r+1)$-cycle in $S_{r+1} \cong W(\Phi)$

A Coxeter element c has no nonzero fixed points in $\mathbb{R}^r$. Given a nonzero $v \in \mathbb{R}^r$, $\exists\, c^j(v)$ which is not positive (otherwise $\sum_{j=0}^{r-1} c^j(v)$ would be a nonzero fixed vector in $\mathbb{R}^r$).

have a basis, and says that not all coefficients can be nonneg.

Definition 8.23: Suppose $1, \dots, r$ is an admissible numbering of the vertices of Q (recall vertex j is a sink for $s_{j+1} \dots s_r Q$). Then the Coxeter functor wrt this numbering is the functor

$$C^+ := S_1^+ \dots S_r^+ : \begin{matrix}\text{reptns}\\ \text{of } Q\end{matrix} \longrightarrow \begin{matrix}\text{reptns}\\ \text{of } Q\end{matrix}$$

(each arrow is being reversed twice)

$$C^- := S_r^- \dots S_1^- : \begin{matrix}\text{reptns}\\ \text{of } Q\end{matrix} \longrightarrow \begin{matrix}\text{reptns}\\ \text{of } Q\end{matrix}$$

2 things to notice: (1) S_r^+ : rep^{n}s of Q $\to$ rep^{n}s of $s_r Q$ using fact that r is a sink b/c of numbering

$\Rightarrow$ S_1^- : rep^{n}s of Q $\to$ rep^{n}s of $s_1 Q$ using fact that 1 is a source

and we can do this process inductively: j a sink for $s_{j+1} \dots s_r Q$, and j a source for $s_{j-1} \dots s_1 Q$

S_j^+ : rep^{n}s of $s_{j+1} \dots s_r Q \to s_j s_{j+1} \dots s_r Q$

similarly S_j^- : rep^{n}s of $s_{j-1} \dots s_1 Q \to s_j s_{j-1} \dots s_1 Q$

Lemma 8.24: Given an indecomposable representation V of Q, either

(i) $C^- C^+(V) = V$, or

(ii) $C^+(V) = 0$

In case (i), the effect of doing C^+ at the level of dimension vectors, we get

$$\text{dim. vector of } C^+(V) = s_{\alpha_1} \dots s_{\alpha_r}\left(\begin{matrix}\text{dim. vector}\\ \text{of } V\end{matrix}\right)$$

proof: We get case (ii) if any of the $S_{j+1}^+ \dots S_r^+(V)$ is the 1-dimensional representation concentrated at vertex j. Then applying S_j^+ gives zero. Otherwise we are in case (i). $\square$

Case (ii) makes sense: remember applying S_j^+ gives us a rep^n W of $s_j \dots Q$ with $W_i = V_i$ $i \neq j$, and $W_j = \oplus \ker(\varphi)$ where φ are the maps going $\varphi: V_i \to V_j$ representing directed edge $i \to j$. So then $W_j \subseteq \bigoplus_{\substack{i: i \to j \\ i \in \dots Q}} V_i$, and if the previous rep^n has $V_i = 0$ $\forall i \neq j$, then W has $W_i = 0$ $\forall i$, so $W = 0$.

otherwise keep on going (I guess using surjectivity). can recover V using S^{-1} functors, so we have to have that $C^- C^+(V) = V$.

Theorem 8.12 (Gabriel) Q quiver, underlying Coexeter graph which is simply laced + positive definite. ie. A_r, D_r, E_6, E_7, E_8. Then

$$\left\{\begin{array}{l}\text{Isomorphism classes of finite} \\ \text{dimensional indecomposable} \\ \text{representations of } Q.\end{array}\right\} \longleftrightarrow \{\text{positive roots in } \mathbb{R}^r\}$$

the isomorphism classes of indecomposable representations $\leftrightarrow$ positive roots in $\mathbb{R}^r$ wrt. $\Delta = \{\alpha_1, \dots, \alpha_r\}$

$$M \longrightarrow \sum \underbrace{k_i}_{\dim(M_i)} v_i \longleftrightarrow \sum k_i \underbrace{\alpha_i}_{\text{simple roots.}}$$

Proof of Theorem 8.12 (Bernstein - Gelfand - Ponamarev 1972)

Choose an admissable numbering for the vertices of Q. Let C^+ be the corresponding Coxeter functor sending repns of $Q \to$ repns of Q. Let c be the corresponding Coxeter element, $s_{\alpha_1} \cdots s_{\alpha_r} \in W(\Phi)$. Suppose we have an indecomposable repn of Q, with dimension vector $\underline{v}$. remember the vertices arise from the roots

From the above, there is some $m \geq 1$ such that $c^m(\underline{v})$ is not positive. So by 8.24, $(C^+)^m(V) = 0$.

The idea is that C^+ sends rep^{n}s of Q to rep^{n}s of Q, and in particular,

$$\begin{aligned}\text{dimension vector of } C^+(V) &= s_{\alpha_1} \dots s_{\alpha_r}(\text{dim vector of } V) \\ &= s_{\alpha_1} \dots s_{\alpha_r}(\underline{v}) \\ &= c(\underline{v})\end{aligned}$$

generalising, dimension vector of $(C^+)^m(V) = c^m(\underline{v})$

Dimension vectors have by dfn nonnegative coefficients, and so if $c^m(\underline{v}) \leq 0$, $\Rightarrow c^m(\underline{v}) = 0$. So dimension vector of $(C^+)^m(V) = 0 \Rightarrow (C^+)^m(V) = 0$.

Choose m as small as possible with $(C^+)^m(V) = 0$. Thus for some minimal j,

$$S_{j+1}^+ \cdots S_r^+ (C^+)^{m-1}(V) \neq 0$$

0 happens somewhere in the last C^+.

But $S_j^+ S_{j+1}^+ \cdots S_r^+ (C^+)^{m-1}(V) = 0$ smallest possible condition.

Thus by 8.20, $S_{j+1}^+ \cdots S_r^+ (C^+)^{m-1}(V) =$ 1-diml repn concentrated at vertex j,

and

$$V = (C^-)^{m-1} S_r^- \cdots S_{j+1}^- (\text{1-diml repn concentrated at vertex } j).$$

$\underline{v}_j$ = dim. vector for 1-dim rep^n with $v_j = K$ and $v_i = 0\ \forall i \neq j$.

Thus the dimension vector $\underline{v}$ of V is $\underbrace{c^{-m+1} s_{\alpha_r} \cdots s_{\alpha_{j+1}}(\underline{v}_j)}_{\in \Phi^+}$, a positive root. $\rightarrow$ this is a dimension vector so has to be positive (and is non trivial)

Hence, we've shown that if we have an indecomposable rep^n of Q, then the dimension vector of the rep^n gives us a positive root $\in \Phi^+$ by thinking about the Coxeter functors and the Coxeter element associated to the admissible numbering of Q. Isomorphic rep^{n}s give the same dimension vector, so we get the first direction of the correspondence

$$\left\{\begin{array}{l}\text{Isomorphism classes} \\ \text{of finite dimensional} \\ \text{indecomposable rep}^n\text{s} \\ \text{of } Q\end{array}\right\} \longrightarrow \{\text{positive roots}\}$$

Note that the argument shows that any indecomposable repn with the same dimension vector is actually isomorphic to V.

Conversely, if V is a positive root, then some $c^m(\underline{v})$ is not positive. Choose the shortest expression of the form $s_{\alpha_j} \cdots s_{\alpha_r} c^{m-1}(\underline{v})$ to be not positive. But $s_{\alpha_{j+1}} \cdots s_{\alpha_r} c^{m-1}(\underline{v})$ is positive. Thus, $s_{\alpha_{j+1}} \cdots s_{\alpha_r} c^{m-1}(\underline{v}) = \underline{v}_j$ (using 5.17 d), a simple reflection s_α permutes the positive roots $\neq \alpha$ and sends $\alpha \to -\alpha$).

So $(e^-)^{m-1} \mathcal{G}_r^- \cdots \mathcal{G}_{j+1}^- \left(\begin{smallmatrix} \text{1-dim'l repn} \\ \text{concentrated at} \\ \text{vertex } j \end{smallmatrix} \right)$ gives an indecomposable representation of dimension vector $\underline{v}$

$\square$